

# Structure of Exact Renormalization Group Equations for field theory

C. Bervillier

*Laboratoire de Mathématiques et Physique Théorique,  
UMR 7350 (CNRS),  
Fédération Denis Poisson,  
Université François Rabelais,  
Parc de Grandmont, 37200 Tours, France*

---

## Abstract

It is shown that exact renormalization group (RG) equations (including rescaling and field-renormalization) for respectively the scale-dependent *full* action  $S[\tilde{\phi}, t]$  and the scale-dependent *full* effective action  $\Gamma[\tilde{\Phi}, t]$  –in which  $t$  is the “RG-time” defined as the logarithm of a running momentum scale– may be linked together by a Legendre transformation as simple as  $\Gamma[\tilde{\Phi}, t] - S[\tilde{\phi}, t] + \tilde{\phi} \cdot \tilde{\Phi} = 0$ , with  $\tilde{\Phi}(x) = \delta S[\tilde{\phi}] / \delta \tilde{\phi}(x)$  (resp.  $\tilde{\phi}(x) = -\delta \Gamma[\tilde{\Phi}] / \delta \tilde{\Phi}(x)$ ), where  $\tilde{\phi}$  and  $\tilde{\Phi}$  are dimensionless-renormalized quantities. This result, in which any explicit reference to a “cutoff procedure” is absent, makes sense in the framework of field theory. It may be compared to the dimensional regularization of the perturbative field theory, in which the running momentum scale is a pure scale of reference and not a momentum cutoff. It is built from the Wilson historic first exact RG equation in which the field-renormalization step is realized via an operator which is redundant and exactly marginal at a fixed point, the properties of which are conserved by the Legendre transformation and which modifies the usual removal of the overall UV cutoff  $\Lambda_0$  by associating it with the removal of an overall IR cutoff  $\mu$ . Because the final equations do not refer to any true cutoff (even for the scale-dependent  $\Gamma$ ), it reinforces the idea that one may get rid of the achronistic procedure of “regularizing” the theory via an explicit “cutoff function”, procedure which is often seen as an inconvenience to treat modern problems in field theory.

*Key words:* Exact renormalization group equation, Reparametrization invariance, Exactly marginal redundant operator, Anomalous dimension

*PACS:* 05.10.Cc, 11.10.Gh, 11.10.Hi, 64.60.ae

---



---

*Email address:* `claud.bervillier@lmpt.univ-tours.fr` (C. Bervillier).

## 1 Introduction

There are two families of exact<sup>1</sup> renormalization group (RG) equations (ERGE) according to whether one considers a Wilsonian ERGE [1] –for the scale-dependent action  $S_\Lambda[\phi]$ – or a Wetterichian ERGE –for the scale-dependent effective action  $\Gamma_k[M]$  (also named “effective average action” [2]). The two scales  $\Lambda$  and  $k$  have the following meanings. When usually referring to an action  $S[\phi]$  and an effective action  $\Gamma[M]$ ,  $\phi(x)$  represents a scalar field attached to the microscopic description of a system (like the spin for the Ising model) and  $M(x)$  a macroscopic quantity, averaged over the whole volume (like the magnetization). There are, implicitly, two fixed and different scales of reference (microscopic and macroscopic). Instead, with the RG theory, these scales become continuously variable:  $\phi(x)$  and  $M(x)$  are both averages over *partial* volumes of the system, then  $\phi(x)$  is attached to an ultraviolet (UV) momentum scale  $\Lambda$  (“short distances”), whereas  $M(x)$  is attached to an infrared (IR) momentum scale  $k$  (“large distances”). Despite their different names, and because they are not fixed, the two scales  $k$  and  $\Lambda$  may well be chosen to be equal for a given system. So, in the following we set  $k = \Lambda$ .

The RG transformation involves the following three (RG-)steps<sup>2</sup>:

- RG-step 1 A reduction of the degrees of freedom (decimation), that consists of an integration of the high momentum components of the field, generating a “scale-dependent action” with a reduced “running” scale  $\Lambda \rightarrow \mu < \Lambda$ ;
- RG-step 2 A (classical) rescaling of the momenta back to the “initial value of the running scale”,  $\mu \rightarrow \Lambda$ , that is conveniently implemented by imposing that any dimensioned quantity is rendered dimensionless by means of classical powers of  $\Lambda$  (such as  $q = \Lambda \tilde{q}$ , for a momentum).
- RG-step 3 A field-renormalization<sup>3</sup>  $\phi(x) \rightarrow \sqrt{Z_3(\mu/\Lambda)} \phi(x)$  (equivalently  $M(x) \rightarrow$

---

<sup>1</sup> This is the historic appellation chosen in order to distinguish these equations from approximate realizations of the RG steps. Some authors prefer naming them non-perturbative or functional RG equations. However, the term “exact” has the merit of clearly designing the subject of interest. Moreover, nobody would have the idea of changing the historic appellation “Renormalization Group” on the grounds that the mathematical notion of group does not play an essential role in the subject.

<sup>2</sup> In fact, adding the possibility of redefining the field at will provided that the quasi-locality of the action is not destroyed (see remark 3), there are four steps allowed (as the three musketeers were in fact four).

<sup>3</sup> It is important to notice that, in the current literature, the field-renormalization function  $Z_3$  is most often presented as a function  $Z_3(\ell)$ , with  $\ell = \Lambda/\mu_0 < 1$  in which  $\mu_0$  is an arbitrary momentum scale introduced for convenience to define the RG-time  $t$  [see eq. (3)]. In that case the dependence on the running scale  $\Lambda$  within  $Z_3$  is inverse of the present case with  $\mu/\Lambda < 1$ . This is because, the current use of  $Z_3$  in the construction of an ERGE as a *global* field-renormalization (i.e., over

$\sqrt{Z_3(\mu/\Lambda)} M(x))$  that removes an indetermination linked to the reparametrization invariance and potentially introduces the anomalous dimension of the field  $\eta^*$  in the vicinity of a fixed point (see appendix A).

In principle, an ERGE expresses the evolution of the action (or of the effective action) under an *infinitesimal* realization of these three RG-steps.

Most often one refers to the RG transformation through RG-step 1 exclusively and one roughly defines an ERGE to be the evolution of  $S_\Lambda[\phi]$  (resp.  $\Gamma_\Lambda[M]$ ) under an infinitesimal change of a momentum cutoff  $\Lambda$  artificially introduced within the actions; the two other RG-steps are usually seen to be secondary and often put into a single step. Actually, we show in this article, that RG-step 3 may be extremely helpful in simplifying the relationship between the two families of ERGE.

One already knows that a Legendre transformation [3–8] links together RG flow equations for *truncations* of both  $S_\Lambda[\phi]$  and  $\Gamma_\Lambda[M]$  [noted hereafter respectively  $S_{\text{int},\Lambda}[\phi]$  and  $\Gamma_{\text{int},\Lambda}[M]$ , see (18) and (39)]. But this Legendre transformation does not account for RG-steps 2 and 3. When these latter steps are accounted for, the actions become functions of dimensionless (and renormalized) quantities noted respectively  $S[\tilde{\phi}, t]$  and  $\Gamma[\tilde{M}, t]$  (similarly  $S_{\text{int}}[\tilde{\phi}, t]$  and  $\Gamma_{\text{int}}[\tilde{M}, t]$ , see section 2.1) in which  $t$  is a dimensionless measure of  $\Lambda$  called the RG-time defined by (3). It is a matter of fact that the transformation which links  $S_{\text{int}}[\tilde{\phi}, t]$  and  $\Gamma_{\text{int}}[\tilde{M}, t]$  becomes horribly complicated when the field-renormalization RG-step 3 is accounted for in an “ordinary way”, with first order differential equations to be solved in which the anomalous dimension  $\eta^*$  is very involved [6–8].

In the present paper we show the existence of a Legendre transformation between the *full* scale-dependent action  $S[\tilde{\phi}, t]$  and the *full* scale-dependent effective action  $\Gamma[\tilde{\Phi}, t]$  which is as simple as:

$$\Gamma[\tilde{\Phi}, t] - S[\tilde{\phi}, t] = -\tilde{\phi} \cdot \tilde{\Phi}, \quad (1)$$

$$\tilde{\Phi}(\tilde{x}) = \frac{\delta S[\tilde{\phi}, t]}{\delta \tilde{\phi}(\tilde{x})}, \quad (2)$$

and which allows to readily deduce one ERGE from the other. This result is made possible by a particular realization of the field-renormalization RG-step 3, similar to that originally utilized by Wilson [1]. It relies on the use of an

---

the finite range  $[\Lambda, \mu_0]$  with  $\Lambda < \mu_0$ ) is introduced as a convenient add-on to the realization of the RG-steps that are instead themselves effected independently and infinitesimally via the decreasing  $\Lambda \rightarrow \Lambda - d\Lambda$  (see section 2.4.4).

operator, which is redundant and exactly marginal at fixed points (EMRO), the properties of which are conserved by the Legendre transformation and which modifies the consequences of the usual process of sending to infinity the overall (initial) UV cutoff  $\Lambda_0 > \Lambda$ .

An important consequence of (1, 2), is that the fundamental structural properties of the ERGE [9,10] usually expressed exclusively on  $S_\Lambda [\phi]$  may be directly transposed to  $\Gamma_\Lambda [M]$ .

In particular, it is known [9,10] that the general structure of an ERGE for  $S_\Lambda$  is a consequence of the invariance of the partition function under an infinitesimal field redefinition (see also [11] for the inclusion of the rescaling and field renormalization steps). Indeed, the following infinitesimal change of field definition:

$$\tilde{\phi}'_{\tilde{q}} = \tilde{\phi}_{\tilde{q}} - \psi_{\tilde{q}}(\tilde{\phi}) dt ,$$

implies the following general expression for a Wilsonian RG flow equation:

$$\frac{d}{dt} S[\tilde{\phi}, t] = \int_{\tilde{q}} \left[ \psi_{\tilde{q}}(\tilde{\phi}) \frac{\delta S[\tilde{\phi}, t]}{\delta \tilde{\phi}_{\tilde{q}}} - \frac{\delta \psi_{\tilde{q}}(\tilde{\phi})}{\delta \tilde{\phi}_{\tilde{q}}} \right] ,$$

in which  $\int_q \equiv \int \frac{d^d q}{(2\pi)^d}$  and  $\tilde{\phi}_{\tilde{q}}$  stands for the Fourier component of the field  $\tilde{\phi}(\tilde{x})$  (for the writing conventions used in this article, see section 2.1 and [12])

Now, because it follows from (1, 2) that:

$$\frac{d}{dt} S[\tilde{\phi}, t] = \frac{d}{dt} \Gamma[\tilde{\Phi}, t] ,$$

then one readily obtains the general expression of the ERGE for  $\Gamma[\tilde{\Phi}, t]$ :

$$\begin{aligned} \frac{d}{dt} \Gamma[\tilde{\Phi}, t] &= \int_{\tilde{q}} \left[ \psi_{\tilde{q}}(\tilde{\phi}) \tilde{\Phi}_{-\tilde{q}} + \frac{1}{\Gamma^{(2)}(\tilde{q}^2; \tilde{\Phi})} \frac{\delta \psi_{\tilde{q}}(\tilde{\phi})}{\delta \tilde{\Phi}_{\tilde{q}}} \right] , \\ \tilde{\phi}(\tilde{x}) &= - \frac{\delta \Gamma[\tilde{\Phi}, t]}{\delta \tilde{\Phi}(\tilde{x})} , \end{aligned}$$

which may be seen as the consequence of the infinitesimal field redefinition:

$$\tilde{\Phi}'_{\tilde{q}} = \tilde{\Phi}_{\tilde{q}} + \frac{1}{\Gamma^{(2)}(\tilde{q}^2; \tilde{\Phi})} \psi_{\tilde{q}}(\tilde{\phi}) dt .$$

Another very interesting property of (1, 2) is the absence of explicit reference to any cutoff function. This enables us to envisage getting completely rid

of the anachronistic necessity of introducing explicitly within the actions a “regularization procedure” by a true momentum cutoff which is often seen as an inconvenience in the framework of field theory.

The organization of the paper is as follows.

In part 2, we present a review of the various structures of ERGE (with a smooth cutoff) that are encountered in the literature. We emphasize both the way RG-step 3 (field-renormalization) is commonly implemented and the resulting form of the Legendre transformation that links the two families of ERGE. This part is divided into sections. We begin by a presentation of our notations (section 2.1) in which the definition and properties of the field-renormalization function  $Z_3$  are recalled. In section 2.2 we present the historic first ERGE proposed by Wilson [1]. We recall the simple form of the associated EMRO and put forward its likely role in the original construction of that ERGE together with the absence of any reference to an explicit UV-cutoff. In section 2.3, we present the original Polchinski version [13] written for a truncation  $S_{\text{int},\Lambda}[\phi]$  of  $S_\Lambda[\phi]$  and without any consideration of RG-steps 2 and 3. Section (2.3.3) shows how these latter two steps have been currently implemented to give the so-called “modified” Polchinski ERGE [14]. Once reexpressed as a RG flow equation for the full action  $S[\tilde{\phi}, t]$ , this leads to a “modified” Wilsonian ERGE [14] for which the EMRO determined in [15] takes on a complicated form (section 2.3.4). In section 2.4 we briefly recall how the ERGE for a truncation  $\Gamma_{\text{int},\Lambda}[M]$  of the effective action  $\Gamma_\Lambda[M]$  –most currently known as the Wetterich ERGE (for a review, see [16])–, may be obtained from the Polchinski ERGE via a Legendre transformation [3, 4]. We then underline the extreme complexity of the most elaborated Legendre transformation (between  $S_{\text{int}}[\tilde{\phi}, t]$  and  $\Gamma_{\text{int}}[\tilde{M}, t]$ ) found so far after implementing RG-step 3 ordinarily [6–8] (section 2.4.5).

In part 3 we propose a structural method for determining the forms of the various kinds of ERGE listed in part 2 so as to maintain simple relations between them. The method, inspired by Wilson’s procedure of implementing RG-step 3, is described in section 3.2. It is based on the realization of RG-step 3 in the ERGE via an EMRO the expression of which is extended out of the vicinity of fixed points. Since the notion of EMRO (seen as a zero-eigenvalue “operator”) exists for any ERGE, we may look at the evolution of one particular EMRO through the various kinds of ERGE. The natural starting point is the Wilson ERGE “*extended to an arbitrary cutoff function*”<sup>4</sup> and its associated still simple EMRO (section 3.3). A new analytical derivation of this equation is given in section 3.3.2 where it is shown that an effective IR-cutoff  $\mu$  appears which is the natural counter part of an overall UV-cutoff  $\Lambda_0$ . Then

---

<sup>4</sup> This is somewhat a misleading expression because there is no actual cutoff in a Wilsonian ERGE.

it is argued that the recourse to an EMRO necessitates the renormalization of the field over all the scales and this implies to having performed the usual limit  $\Lambda_0 \rightarrow \infty$  that automatically induces the limit  $\mu \rightarrow 0$  or reciprocally (section 3.3.3). We then look at the RG-flows for  $S_{\text{int}}[\tilde{\phi}, t]$  and for  $\Gamma_{\text{int}}[\tilde{M}, t]$  while keeping the natural simple expression of the Legendre transformation between  $S_{\text{int}}[\tilde{\phi}, t]$  and  $\Gamma_{\text{int}}[\tilde{M}, t]$  (section 3.4). After having recalled the conditions under which the usual UV limit  $\Lambda_0 \rightarrow \infty$  is justified (vicinity of a fixed point), we show that the associated IR limit  $\mu \rightarrow 0$  is valid if  $\eta^* < 2$ , a usual condition for a fixed point to be a “critical fixed point” [17] (section 3.4.3). The latter IR limit modifies the relation between the IR and UV cutoff functions so that a direct and very simple Legendre transformation then relies the Wilson ERGE “*extended to an arbitrary cutoff function*” (for the *full* action  $S[\tilde{\phi}, t]$ ) to the RG-flow equation for the *full* effective action  $\Gamma[\tilde{M}, t]$  that has been structurally constructed. We conclude in section 4. Because the field-renormalization plays an essential role in our discussion we also recall, in appendix A, how and why  $Z_3$  is related to the anomalous dimension of the field  $\eta^*$ . In a second appendix we present in greater detail than in [18] and on two examples, an adaptation of the procedure of O’Dwyer and Osborn [15] for determining an EMRO for a given Wilsonian ERGE.

## 2 Summary of previous episodes

In order to well expose the issue discussed in this article it is useful to first present a brief summary of previous episodes of the ERGE history. Several modern reviews or lectures on the subject are available in the literature [12, 16, 17] [19]– [35].

### 2.1 Notations

Let us first introduce some of our notation conventions.

The scale-dependent action  $S_\Lambda[\phi]$  and the scale-dependent effective action  $\Gamma_\Lambda[M]$  will be noted respectively  $S[\tilde{\phi}, t]$  and  $\Gamma[\tilde{M}, t]$  in which:

$$t = -\ln(\Lambda/\mu_0) , \quad (3)$$

is the (dimensionless) RG-time,  $\mu_0$  being an arbitrary momentum-scale of reference larger than  $\Lambda$  and which is different from the overall cutoff  $\Lambda_0$  (that will finally be sent to infinity)

The rescaling step of the RG procedure (RG-step 2) may be accounted for

by expressing the dimensions via factorized powers of  $\Lambda$ . In the following we denote dimensionless quantities by letter with an upper tilde, such as, for example:

$$q = \Lambda \tilde{q}, \quad (4)$$

$$\phi(x) = \Lambda^{d_\phi^{(c)}} \tilde{\phi}(\tilde{x}), \quad (5)$$

in which  $d_\phi^{(c)}$  is the classical dimension of the field ( $\phi(x)$  or  $M(x)$ ). In order to make contact with the historic first version of the ERGE [1] we assume the unusual value (see footnote 9):

$$d_\phi^{(c)} = \frac{d}{2} - n_0, \quad (6)$$

where  $d$  is the spatial dimension and  $n_0$  will take on the value 0 or 1 according to the choice of the cutoff function [see eq. (19)]. We shall show that Wilson's choice:

$$d_\phi^{(cw)} = \frac{d}{2}, \quad (7)$$

corresponding to  $n_0 = 0$ , is linked to the possibility of getting rid of the regularization procedure via field-redefinitions.

We will use also the following scale-dependent “anomalous” dimensions:

$$d_\phi^{(\pm)}(t) = \frac{d}{2} \pm \varpi_0(t), \quad (8)$$

$$\varpi_{n_0}(t) = 1 - n_0 - \frac{\eta(t)}{2}, \quad (9)$$

in which the function  $\eta(t)$  is determined so as to keep constant (i.e. independent of  $t$ ) the coefficient of one term<sup>5</sup> of  $S$ . At a fixed point,  $\eta(t)$  is a constant  $\eta^*$  called the anomalous dimension of the field (see appendix A).

Sometimes, when no confusion may arise with the usual actions, we lightly refer to the scale-dependent actions as  $S$  and  $\Gamma$ . When needed, we still refer to dimensioned quantities as previously ( $\Lambda, \phi, x, q, M$ , etc.).

We also use sometimes the following writing conventions:

---

<sup>5</sup> This step is required due to the reparametrization invariance that implies that the flow of one coefficient of  $S$  is redundant. To eliminate this redundancy, and to set the momentum scales, one usually keeps constant the coefficient of the kinetic term along the flow of  $S$ .

- $A_q, B(q^2)$  are Fourier transformed of respectively  $A(x), B(x, y)$  when the usual invariances by translation and rotation in space are assumed.
- $A \cdot C$  stands for  $\int d^d x A(x) C(x)$  or  $\int_q A_q C_{-q}$
- $A \cdot B \cdot C$  stands for  $\int d^d x d^d y A(x) B(x, y) C(y)$  or  $\int_q A_q B(q^2) C_{-q}$ .

For the sake of an easy understanding of the present article it is important to recall that the field-renormalization function  $Z_3(\ell)$  is usually utilized to define the renormalized field  $\phi_R$  as follows<sup>6</sup>:

$$\phi = \sqrt{Z_3(\ell)} \phi_R, \quad \text{with} \quad \ell < 1.$$

Most often, to simplify the writing, we shall drop the subscript “ $R$ ” when no confusion may arise.

So defined,  $Z_3(\ell)$  is linked to the anomalous dimension of the field  $\eta(t)$  as (see appendix A):

$$\ell \frac{d}{d\ell} Z_3(\ell) = -2 \varpi_{n_0}(t) Z_3(\ell), \quad (10)$$

in which  $n_0$  is defined in eq. (19) in order to account for different versions of ERGE. In (10) we have intentionally distinguished  $\ell$  from  $e^{-t}$  with  $t$  defined by (3); this distinction has some importance as discussed in section 3.3.2.

## 2.2 The historic Wilson ERGE

The historic first ERGE [1] has been set up directly for the scale-dependent full action  $S[\tilde{\phi}; t]$ , including rescaling and field-renormalization, it may be written under the following form [18]:

$$\begin{aligned} \dot{S}[\tilde{\phi}; t] = & \int_{\tilde{q}} 2\tilde{q}^2 \left( \frac{\delta^2 S}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} + \tilde{\phi}_{\tilde{q}} \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \right) \\ & + \mathcal{G}_{\text{dil}}(S, \tilde{\phi}, d_{\phi}^{(cw)}) + \varpi_0(t) \mathcal{O}_w(S, \tilde{\phi}), \end{aligned} \quad (11)$$

in which  $\dot{S}[\tilde{\phi}; t]$  stands for  $dS[\tilde{\phi}; t]/dt|_{\tilde{\phi}}$ .

The first line of (11) corresponds to a realization of the RG-step 1 (decimation) by means of an “incomplete integration”.

The second line shows two parts:

---

<sup>6</sup> Wetterich’s field-renormalization function  $Z_\Lambda$  is the inverse of  $Z_3$  (see section 2.4.4) but this is a matter of convention provided that the definition of the anomalous dimension  $\eta(t)$  be unchanged (see appendix A).



(1) the rescaling part (RG-step 2):

$$\mathcal{G}_{\text{dil}}(S, \tilde{\phi}, d_\phi) = \int_{\tilde{q}} \left[ (d - d_\phi) \tilde{\phi}_{\tilde{q}} + \tilde{\mathbf{q}} \cdot \frac{\partial}{\partial \tilde{\mathbf{q}}} \tilde{\phi}_{\tilde{q}} \right] \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}}, \quad (12)$$

in which  $d_\phi = d_\phi^{(cw)}$  is the classical dimension of the field, it is given by (7) –i.e. (6) with<sup>7</sup>  $n_0 = 0$ .

(2) the field-renormalization part (RG-step 3) with:

$$\mathcal{O}_w(S, \tilde{\phi}) = \int_{\tilde{q}} \left( \frac{\delta^2 S}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} + \tilde{\phi}_{\tilde{q}} \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} - 1 \right). \quad (13)$$

By separating the contribution proportionnal to  $\mathcal{O}_w(S, \tilde{\phi})$  in (11), we want to highlight the following ideas that, presumably, are at the source of Wilson's particular implementation of RG-step 3.

At a fixed point  $S^*$  of (11),  $\eta(t)$  takes on the constant value  $\eta^*$  and  $\mathcal{O}_w(S^*, \tilde{\phi})$  is an EMRO responsible for an (infinitesimal) generation of a line of equivalent fixed points under the change  $\tilde{\phi} \rightarrow \tilde{\phi}(1 + \delta a)$ ,  $\delta a$  being a pure (infinitesimal) constant independent of  $t$ . Indeed, as defined in (13),  $\mathcal{O}_w(S, \tilde{\phi})$  is the infinitesimal version of the transformation<sup>8</sup>  $U_\kappa$ , introduced by Bell and Wilson [36] (see also [37]) that commutes with the RG procedure and is associated to a change of  $\tilde{\phi}$  into  $\tilde{\phi}' = \kappa \tilde{\phi}$ :

$$U_\kappa \exp(-S[\tilde{\phi}]) \propto \int \mathcal{D}\tilde{\phi} \exp \left\{ -\frac{1}{2(1-\kappa^2)} \int_q |\kappa \tilde{\phi}_q - \tilde{\phi}'_q|^2 - S[\tilde{\phi}] \right\}. \quad (14)$$

For  $\kappa$  infinitesimally close to unity, the properties of the Gaussian integral yields the expression (13).

As any redundant operator  $\mathcal{O}(S, \tilde{\phi})$  expresses also under the following form [9, 10]:

$$\mathcal{O}(S, \tilde{\phi}) = \int_{\tilde{q}} \left[ \psi_{\tilde{q}}(\tilde{\phi}, S) \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} - \frac{\delta \psi_{\tilde{q}}(\tilde{\phi}, S)}{\delta \tilde{\phi}_{\tilde{q}}} \right], \quad (15)$$

then for  $\mathcal{O}_w(S, \tilde{\phi})$  given by (13) we have:

$$\psi_{\tilde{q}}^{(w)}(\tilde{\phi}, S) = \tilde{\phi}_{\tilde{q}} - \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}}. \quad (16)$$

Contrary to what one might think, the expression of an EMRO is not necessarily linked to  $U_\kappa$  because that expression depends on conventions and on

<sup>7</sup> Thecnically, the form of the first line of (11) implies that  $n_0 = 0$  [6].

<sup>8</sup> Here  $\kappa$  may depend on  $t$ .

the way both the field-renormalization and the RG-step 1 are realized (e.g., see appendix B).

In the following we shall call Wilsonian EMRO an EMRO formed by a linear combination of  $\tilde{\phi}_{\tilde{q}}$  and  $\frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}}$  such as:

$$\psi_{\tilde{q}}^{(wil)}(\tilde{\phi}, S) = \bar{a}(\tilde{q}^2) \tilde{\phi}_{\tilde{q}} + \bar{b}(\tilde{q}^2) \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} . \quad (17)$$

**Remark 1** *It is to be noticed that, in the procedure described in [1], there is no overall UV-cutoff  $\Lambda_0$  explicitly considered. The procedure is so that the fields with large  $q$  are merely more integrated than those with small  $q$  (without having to refer to any explicit existence of a cutoff  $\Lambda_0$  that would distort the action  $S$ ). Only an initial RG-time  $t = 0$  is defined at which the “incomplete integration” is started. It is like the introduction of the formal scale  $\mu_0$  in the definition (3) of  $t$ , which never appears explicitly in the ERGE. Any advised user of ERGE knows that the “bare action” is merely a simple form of an action chosen at will at  $t = 0$  as initial condition to the ERGE without having to specify any regularization process. Thus, to use a common language, in the Wilson ERGE (11), the limit  $\Lambda_0 \rightarrow \infty$  is implicitly performed .*

**Remark 2** *In conjunction with the preceding remark and for later use (see section 3.3.2), it is interesting to recall here (in other terms) a remark of Bell and Wilson [36]: this way of renormalizing the field (using  $U_\kappa$  and thus an EMRO), affects “every old” fields  $\phi$  in contrast with the reduction of variables effected by RG-step 1 that affects only a part of them.*

**Remark 3** *The integral over  $\tilde{q}$  of the historic first ERGE (11) is not necessarily well defined for large  $\tilde{q}$  so that a redefinition of the field  $\tilde{\phi}_{\tilde{q}} \rightarrow g(\tilde{q}^2) \tilde{\phi}_{\tilde{q}}$  with  $g(\tilde{q}^2) \rightarrow \infty$  sufficiently rapidly (and  $g(0) = 1$  to preserve the quasi-locality of the action), could be required prior to explicit calculations, e.g., as in the framework of the derivative expansion [38]. This field redefinition is allowed owing to the property of reparametrization invariance and could be added to the list of RG-steps as an optional fourth step that may be implemented at will.*

### 2.3 The Polchinski ERGE

Based on an explicit reference to an arbitrary smooth-UV-cutoff function, Polchinski [13] has constructed a RG flow equation for a truncation of  $S_\Lambda[\phi]$ , denoted here  $S_{\text{int},\Lambda}[\phi]$  and defined as follows:

$$S_\Lambda[\phi] = \frac{1}{2} \phi \cdot P^{-1} \cdot \phi + S_{\text{int},\Lambda}[\phi] , \quad (18)$$

In (18), a quadratic part has been evidenced in order to make explicit (though arbitrary) the UV cutoff function  $P(q^2, \Lambda)$  that, for the sake of comparison with the Wilson procedure recalled in section 2.2, we choose to write under the form:

$$P(q^2, \Lambda) = \frac{K(\tilde{q}^2)}{(q^2)^{n_0}}, \quad (19)$$

in which  $n_0$  may take on the value 0 or 1, and  $K$  is dimensionless with the assumed property that<sup>9</sup>

$$\lim_{\Lambda \rightarrow \infty} K(\tilde{q}^2) = \text{const.} \quad (20)$$

So defined,  $P$  has the dimension  $-2n_0$ .

### 2.3.1 Decimation

RG-step 1, expressed as a variation of  $S_{\text{int}, \Lambda}$  under an infinitesimal change of  $\Lambda$ , leads to what is usually named the Polchinski ERGE [13] (discussing the way it is obtained is out of the scope of the present article, readers interesting by that issue may look also at [43, 44]):

$$-\Lambda \frac{\partial}{\partial \Lambda} S_{\text{int}, \Lambda} \Big|_{\phi} = \frac{1}{2} \int_q \Lambda \frac{\partial}{\partial \Lambda} P(q^2, \Lambda) \Big|_q \left[ \frac{\delta^2 S_{\text{int}, \Lambda}}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S_{\text{int}, \Lambda}}{\delta \phi_q} \frac{\delta S_{\text{int}, \Lambda}}{\delta \phi_{-q}} \right]. \quad (21)$$

Coming back to the full action  $S_{\Lambda}[\phi]$  via (18), (21) gives (up to an additive constant term which is currently neglected in the framework of field theory):

$$-\Lambda \frac{\partial}{\partial \Lambda} S_{\Lambda} \Big|_{\phi} = \frac{1}{2} \int_q \Lambda \frac{\partial}{\partial \Lambda} P(q^2, \Lambda) \Big|_q \left[ \frac{\delta^2 S_{\Lambda}}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S_{\Lambda}}{\delta \phi_q} \frac{\delta S_{\Lambda}}{\delta \phi_{-q}} + 2P^{-1} \phi_q \frac{\delta S_{\Lambda}}{\delta \phi_q} \right].$$

---

<sup>9</sup> Polchinski has in fact chosen a dimensioned form of  $P$  namely:  $P(q^2, \Lambda) = K \left( \frac{q^2}{\Lambda^2} \right) / (q^2 + m^2)$  with  $K$  dimensionless and, thus, with a classical dimension  $d_{\phi}^{(c)} = (d - 2)/2$  for the field, i.e. with  $n_0 = 1$ . In that case:  $\lim_{\Lambda \rightarrow \infty} P(q^2, \Lambda) \propto \frac{1}{q^2 + m^2}$ . The idea behind this choice is reminiscent of the old perturbative version of the RG, in which the cutoff  $\Lambda$  is seen as an artificial parameter that modifies the propagator only temporarily. In the perturbative view, the scale  $\Lambda$  is introduced “by hand” (and this arbitrariness is intended to be finally “washed out” in the final theory). This is in contrast to Wilson’s view in which  $\Lambda$  is seen as an integral parameter of the RG theory.

### 2.3.2 Rescaling

The implementation of RG-step 2 (rescaling step) introduces dimensionless quantities thus we use dimensionless field  $\tilde{\phi}$  as defined in (5, 6) and we have:

$$\begin{aligned} \frac{1}{2} \Lambda \frac{\partial}{\partial \Lambda} P(q^2, \Lambda) \Big|_q &= \Lambda^{-2n_0} \left[ -n_0 \tilde{P}(\tilde{q}^2) - \tilde{q}^2 \tilde{P}'(\tilde{q}^2) \right] \\ &= -\Lambda^{-2n_0} (\tilde{q}^2)^{1-n_0} K'(\tilde{q}^2), \end{aligned}$$

where the prime denotes a derivative w.r.t.  $\tilde{q}^2$ . After completing RG-step 2, but not yet RG-step 3 (field-renormalization), we get for the full action:

$$\begin{aligned} \dot{S}[\tilde{\phi}, t] &= - \int_{\tilde{q}} (\tilde{q}^2)^{1-n_0} K'(\tilde{q}^2) \left[ \frac{\delta^2 S}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} + 2\tilde{P}^{-1} \tilde{\phi}_{\tilde{q}} \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \right] \\ &\quad + \mathcal{G}_{\text{dil}}(S, \tilde{\phi}, d_{\phi}^{(c)}). \end{aligned} \quad (22)$$

It is interesting to perform also RG-step 2 on (21). We get:

$$\begin{aligned} \dot{S}_{\text{int}}[\tilde{\phi}, t] &= - \int_{\tilde{q}} (\tilde{q}^2)^{1-n_0} K'(\tilde{q}^2) \left[ \frac{\delta^2 S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{-\tilde{q}}} \right] \\ &\quad + \mathcal{G}_{\text{dil}}(S_{\text{int}}, \tilde{\phi}, d_{\phi}^{(c)}). \end{aligned} \quad (23)$$

It is a simple exercise to verify that this latter equation may be deduced from (22) via the dimensionless version of (18), namely:

$$S[\tilde{\phi}, t] = \frac{1}{2} \tilde{\phi} \cdot \tilde{P}^{-1} \cdot \tilde{\phi} + S_{\text{int}}[\tilde{\phi}, t], \quad (24)$$

that induces the relation

$$\dot{S}[\tilde{\phi}, t] = \dot{S}_{\text{int}}[\tilde{\phi}, t]. \quad (25)$$

If, at this stage of the RG program, we perform the change  $\tilde{\phi} \rightarrow \tilde{\phi} \sqrt{\tilde{P}}$  that implies  $n_0 = 0$  (to preserve quasi-locality) then [by virtue of (12), with  $d_{\phi}^{(c)} \rightarrow d_{\phi}^{(cw)}$  given by (7) and also  $\tilde{P} = K$ ] equation (22) reads:

$$\begin{aligned}
\dot{S} [\tilde{\phi}, t] = & - \int_{\tilde{q}} \tilde{q}^2 \frac{\tilde{P}'(\tilde{q}^2)}{\tilde{P}} \left[ \frac{\delta^2 S}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} + 2 \tilde{\phi}_{\tilde{q}} \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \right] \\
& + \mathcal{G}_{\text{dil}} \left( S, \tilde{\phi}, d_{\phi}^{(cw)} \right) + \int_{\tilde{q}} \tilde{q}^2 \frac{\tilde{P}'(\tilde{q}^2)}{\tilde{P}} \tilde{\phi}_{\tilde{q}} \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}}.
\end{aligned} \tag{26}$$

Comparing with (11) it is easy to verify that Wilson's version corresponds to Polchinski's version (provided the field redefinition into  $\tilde{\phi} \sqrt{\tilde{P}(\tilde{q}^2)}$ ) with the choice  $\tilde{P}(\tilde{q}^2) = e^{-2\tilde{q}^2}$  [39]. One could already notice the difference of spirit induced by this field redefinition: the cutoff function has finally been removed from the action so that  $\Lambda$  appears to be a pure scale  $\Lambda = e^{-t} \mu_0$  (see remark 1).

It is also interesting to notice that in the Polchinski approach, the running cutoff  $\Lambda$  itself is an overall UV-cutoff  $\Lambda_0$ . Consequently, at this stage, it is not obvious to see how it is possible to reproduce in a Polchinski ERGE, the limit  $\Lambda_0 \rightarrow \infty$  at fixed  $\Lambda$  that is commonly performed and discussed in the case of a Wetterichian ERGE (see section 2.4). That important issue is discussed in section 3.4.3.

The correspondence with the Wilson version established just above is not complete since it has been obtained in absence of field-renormalization. Commonly, this latter RG-step is completed following the procedure of Ball et al [14], who did not introduce  $\eta(t)$  the same way as Wilson did to get (11).

### 2.3.3 Field-renormalization: the “modified” Polchinski ERGE

The implementation of RG-step 3 in the flow equation for  $S_{\text{int}} [\tilde{\phi}, t]$  can be regularly realized by replacing in (23)  $d_{\phi}^{(c)}$  by  $d_{\phi}^{(-)}$  as defined in eq. (8) to get:

$$\begin{aligned}
\dot{S}_{\text{int}} [\tilde{\phi}, t] = & - \int_{\tilde{q}} (\tilde{q}^2)^{1-n_0} K'(\tilde{q}^2) \left[ \frac{\delta^2 S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{-\tilde{q}}} \right] \\
& + \mathcal{G}_{\text{dil}} \left( S_{\text{int}}, \tilde{\phi}, d_{\phi}^{(-)} \right).
\end{aligned} \tag{27}$$

Indeed, this procedure amounts to expressing on  $S_{\text{int}}$  the regular infinitesimal field-renormalization associated to an infinitesimal realization of RG-step 1. This way of introducing  $\eta(t)$ , the common way, is said “linear” in [18] by reference to the fact that it contributes linearly in  $S_{\text{int}}$  (or  $S$ ) in the flow equation, in contrast to the Wilson procedure which is “non-linear” in this respect.

In practice<sup>10</sup>, instead of considering  $S_{\text{int}}$ , Ball et al [14] have chosen to perform the above change  $d_\phi^{(c)}$  by  $d_\phi^{(-)}$  within the flow equation of the *full action*  $S[\tilde{\phi}, t]$  (22) which then reads:

$$\begin{aligned} \dot{S}[\tilde{\phi}, t] = & - \int_{\tilde{q}} (\tilde{q}^2)^{1-n_0} K'(\tilde{q}^2) \left[ \frac{\delta^2 S}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} + 2\tilde{P}^{-1} \tilde{\phi}_{\tilde{q}} \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \right] \\ & + \mathcal{G}_{\text{dil}}(S, \tilde{\phi}, d_\phi^{(-)}) . \end{aligned} \quad (28)$$

(Because the implementation of RG-step 3 is different from the original Wilson way that yields (11), we refer to (28) as the “modified” Wilson ERGE extended to an arbitrary cutoff.)

Assuming<sup>11</sup> that the effect of the infinitesimal implementation of RG-step 3 does not affect eq. (24), and, consequently, that the condition (25) is satisfied (see remark 4 below), one finally gets the flow equation of Ball et al for  $S_{\text{int}}$ :

$$\begin{aligned} \dot{S}_{\text{int}}[\tilde{\phi}, t] = & - \int_{\tilde{q}} (\tilde{q}^2)^{1-n_0} K'(\tilde{q}^2) \left[ \frac{\delta^2 S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{-\tilde{q}}} \right] \\ & + \mathcal{G}_{\text{dil}}(S_{\text{int}}, \tilde{\phi}, d_\phi^{(-)}) - \varpi_{n_0}(t) \tilde{\phi} \cdot \tilde{P}^{-1} \cdot \tilde{\phi} . \end{aligned} \quad (29)$$

This equation has been called the “modified” Polchinski flow equation [6–8] because it involves an unusual quadratic term proportionnal to<sup>12</sup>  $\varpi_{n_0}(t)$  as defined in (9).

**Remark 4** *The additional Gaussian term that characterizes the “modified” flow equation [last term of (29)] is due to the fact that the modification of the quadratic term of (18) induced by the field-renormalisation step is not compensated by a scale-dependent counter-part within the cutoff function  $P$ . Its presence in the RG-flow equation is actually due to the arbitrary condition (25). If, instead, we impose the more regular condition:*

$$\dot{S} = \dot{S}_{\text{int}} + \varpi_{n_0}(t) \tilde{\phi} \cdot \tilde{P}^{-1} \cdot \tilde{\phi} , \quad (30)$$

*then the RG-flow equation for  $S_{\text{int}}$  reduces merely to (27) which is also a well acceptable flow equation.*

<sup>10</sup> The reasoning of Ball et al is different but it is more convenient, and equivalent, to present it that way.

<sup>11</sup> This assumption is not obviously justified, see remark 4.

<sup>12</sup> In the version of Ball et al [14] this term is proportionnal to  $\varpi_1(t) = -\eta(t)/2$  due to the choice of a dimensioned cutoff function  $P(q^2, \Lambda)$  implying  $n_0 = 1$ , see footnote 9.

### 2.3.4 Complicated expression of an EMRO for the “modified” Wilson ERGE

Beyond the fact that the original Wilson ERGE is obviously not recovered with the common (linear) way of completing the field-renormalization (what is a priori nothing to be ashamed of), there are important consequences.

In particular, the expression of a Wilsonian EMRO [see eq. (17)] –responsible for the existence of lines of equivalent fixed points– takes on a complicated form with the common procedure [15] whereas this form is simple with the Wilson procedure (see also appendix B).

In order to get the expression of the Wilsonian EMRO associated to the “modified” Wilsonian ERGE (28), O’Dwyer and Osborn [15] have looked for an EMRO, corresponding to  $\psi_{\tilde{q}}^{(wil)}(\tilde{\phi}, S)$  as given by (17).

In the case where the cutoff function is given by (19) with  $n_0 = 1$ , they have obtained<sup>13</sup>:

$$\bar{a}(x) = 1 + x\bar{b}(x), \quad \bar{b}(x) = x^{\eta^*/2-1} K^2(x) \int_0^x u^{-\eta^*/2} \frac{K'(u)}{[K(u)]^2} du. \quad (31)$$

This is in contrast with the Wilson procedure with which the chosen EMRO has the simple form (16) and, once “extended to an arbitrary cutoff function”  $\tilde{P}(\tilde{q}^2)$ , reads (provided that  $n_0 = 0$ ) [18]:

$$\psi_{q,\mathcal{O}} = \tilde{\phi}_q - \tilde{P}(\tilde{q}^2) \frac{\delta S^*}{\delta \tilde{\phi}_{-q}}.$$

In appendix B we determine a similar simple expression of an EMRO in the Polchinski case  $n_0 = 1$  when RG-step 3 is not linearly implemented.

Similarly to the expression of the EMRO given by (17, 31), the Legendre transformation that links  $S_{\text{int}}$  and  $\Gamma_{\text{int}}$  and their respective RG-flow equations has been found to be extremely complicated [6–8] when the field-renormalization step is implemented the common way. From here it is but a short step to believing that with a Wilson-like non-linear implementation of the field-renormalization step (i.e., not linearly and via an EMRO), the Legendre transformation would take on a simpler form. Let us first recall how the RG-flow equation for  $\Gamma_{\text{int}}$  has been commonly treated.

---

<sup>13</sup> The functions  $\bar{a}$  and  $\bar{b}$  are the functions  $a$  and  $b$  of [15] multiplied by  $K(x)$ .

## 2.4 The Wetterich ERGE and the Legendre transformation

We call Wetterich's flow equation, the ERGE for the effective average action  $\Gamma_{\text{int}}$  (for a review, see [16]).

### 2.4.1 Decimation

There are several ways to get this kind of flow equation [2–4, 40]. For our purposes, the most convenient way is to derive it from the Polchinski ERGE via a Legendre transformation [3, 4]. Let us summarize sketchily the main steps of that derivation (for more details see, e.g. [4]).

We start with the scale-dependent action  $S_\Lambda$  as Polchinski did with (18) but, for convenience, we consider an initial (overall) arbitrary cutoff  $\Lambda_0$  that is greater than the “running” cutoff  $\Lambda$  of the preceding sections. This overall momentum scale is implemented by an arbitrary UV cutoff function  $\Delta_0(q^2, \Lambda_0)$ , similar to  $P$  in (18):

$$S_{\Lambda_0}[\phi] = \frac{1}{2} \phi \cdot \Delta_0^{-1} \cdot \phi + S_{\text{int}, \Lambda_0}[\phi] . \quad (32)$$

Based on the property that the partition function associated to  $S_{\Lambda_0}[\phi]$  can be rewritten in terms of two quadratic parts and two fields such as  $\phi = \phi_1 + \phi_2$ , we write:

$$S_{\Lambda_0}[\phi] = \frac{1}{2} \phi_1 \cdot \Delta_1^{-1} \cdot \phi_1 + \frac{1}{2} \phi_2 \cdot \Delta_2^{-1} \cdot \phi_2 + S_{\text{int}, \Lambda_0}[\phi_1 + \phi_2] ,$$

where  $\Delta_1(q^2; \Lambda)$  is an UV cutoff function associated to  $\Lambda$  and  $\Delta_2(q^2; \Lambda, \Lambda_0)$  an IR cutoff function defined as:

$$\Delta_2(q^2; \Lambda, \Lambda_0) = \Delta_0(q^2, \Lambda_0) - \Delta_1(q^2, \Lambda) . \quad (33)$$

Integrating out partially over  $\phi_2$  generates  $S_{\text{int}, \Lambda}[\phi_1]_\Lambda$  which, using a property of the Gaussian integral, expresses as follows:

$$\exp(-S_{\text{int}, \Lambda}[\phi_1]) = \exp\left(\frac{1}{2} \frac{\delta}{\delta \phi_1} \cdot \Delta_2 \cdot \frac{\delta}{\delta \phi_1}\right) \exp\{-S_{\text{int}, \Lambda_0}[\phi_1]\} . \quad (34)$$

A derivative w.r.t.  $\Lambda$  of (34) gives:

$$\Lambda \frac{\partial S_{\text{int}, \Lambda}}{\partial \Lambda} \Big|_{\phi_1} = \frac{1}{2} \int_q \Lambda \frac{\partial \Delta_2(q^2; \Lambda, \Lambda_0)}{\partial \Lambda} \Big|_q \left[ \frac{\delta^2 S_{\text{int}, \Lambda}}{\delta \phi_{1,q} \delta \phi_{1,-q}} - \frac{\delta S_{\text{int}, \Lambda}}{\delta \phi_{1,q}} \frac{\delta S_{\text{int}, \Lambda}}{\delta \phi_{1,-q}} \right] , \quad (35)$$



which is the Polchinski flow equation (21) provided that [41]:

$$\Lambda \frac{\partial}{\partial \Lambda} \Delta_2 = -\Lambda \frac{\partial}{\partial \Lambda} P, \quad (36)$$

which, from (33) is satisfied if

$$P(q^2, \Lambda) = \Delta_1(q^2, \Lambda). \quad (37)$$

Though we may define a scale-dependent *full* action  $S_\Lambda[\phi_1]$  as:

$$S_\Lambda[\phi_1] = \frac{1}{2} \phi_1 \cdot \Delta_1^{-1} \cdot \phi_1 + S_{\text{int}, \Lambda}[\phi_1], \quad (38)$$

the effective Legendre transformation of interest involves  $S_{\text{int}, \Lambda}[\phi_1]_\Lambda$  and a truncated scale-dependent effective action  $\Gamma_{\text{int}, \Lambda}[M_1]$  linked to the full scale-dependent effective action  $\Gamma_\Lambda[M_1]$  as<sup>14</sup>:

$$\Gamma_\Lambda[M_1] = \frac{1}{2} M_1 \cdot \Delta_2^{-1} \cdot M_1 + \Gamma_{\text{int}, \Lambda}[M_1]. \quad (39)$$

The Legendre transformation thus reads [4]:

$$S_{\text{int}, \Lambda}[\phi_1] = \frac{1}{2} (M_1 - \phi_1) \cdot \Delta_2^{-1} \cdot (M_1 - \phi_1) + \Gamma_{\text{int}, \Lambda}[M_1], \quad (40)$$

$$M_1 = \phi_1 - \Delta_2 \cdot \frac{\delta}{\delta \phi_1} S_{\text{int}, \Lambda}[\phi_1]. \quad (41)$$

Then, from (35) and using the properties:

$$\frac{\delta}{\delta \phi_q} S_{\text{int}, \Lambda}[\phi_1] = \frac{\delta}{\delta M_{1,q}} \Gamma_{\text{int}, \Lambda}[M_1], \quad (42)$$

$$\frac{\delta^2 S_{\text{int}, \Lambda}[\phi_1]}{\delta \phi_{1,q} \delta \phi_{1,-q}} = \frac{\Gamma_{\text{int}, \Lambda}^{(2)}[q; M_1]}{1 + \Delta_2(q^2; \Lambda, \Lambda_0) \Gamma_{\text{int}, \Lambda}^{(2)}[q; M_1]}, \quad (43)$$

we obtain the RG equation of interest:

$$\Lambda \frac{\partial}{\partial \Lambda} \Gamma_{\text{int}, \Lambda}[M_1] = \frac{1}{2} \int_q \Lambda \frac{\partial \Delta_2(q^2; \Lambda, \Lambda_0)}{\partial \Lambda} \frac{\Gamma_{\text{int}, \Lambda}^{(2)}[q; M_1]}{1 + \Delta_2(q^2; \Lambda, \Lambda_0) \Gamma_{\text{int}, \Lambda}^{(2)}[q; M_1]}. \quad (44)$$

---

<sup>14</sup> The common use, inherited from the perturbative field theory, assumes a cutoff function that must vanish when  $\Lambda \rightarrow 0$  so that  $\Gamma_{\text{int}, \Lambda}[M_1] \rightarrow \Gamma[M_1]$  in this limit. Here we need not this condition because we assume, following Wilson, that the quadratic term involving the cutoff function is not an artificial part added by hand but an actual integral part of the full scale-dependent effective action  $\Gamma_\Lambda[M_1]$ .

#### 2.4.2 Limit of infinite overall cutoff

To get the genuine Wetterich RG flow equation, the supplementary limit  $\Lambda_0 \rightarrow \infty$  (at fixed  $\Lambda$ ) must be performed in (44) so as to finally obtain, up to a negligible additive constant, the well-known expression:

$$\Lambda \frac{\partial}{\partial \Lambda} \Gamma_{\text{int}, \Lambda} [M_1] = \frac{1}{2} \int_q \Lambda \frac{\partial R(q^2; \Lambda)}{\partial \Lambda} \bigg|_q \frac{1}{R(q^2; \Lambda) + \Gamma_{\text{int}, \Lambda}^{(2)} [q; M_1]}, \quad (45)$$

in which:

$$R(q^2; \Lambda) = \lim_{\Lambda_0 \rightarrow \infty} \Delta_2^{-1}(q^2; \Lambda, \Lambda_0). \quad (46)$$

#### 2.4.3 Rescaling

The rescaling step is easily accounted for by simple dimensional analysis and leads to:

$$\dot{\Gamma}_{\text{int}} [\tilde{M}_1, t] = \int_{\tilde{q}} \frac{(-n_0 \tilde{R}(\tilde{q}^2) + \tilde{q}^2 \tilde{R}'(\tilde{q}^2))}{\tilde{R}(\tilde{q}^2) + \Gamma_{\text{int}}^{(2)} [\tilde{q}; \tilde{M}_1]} + \mathcal{G}_{\text{dil}} (\Gamma_{\text{int}}, \tilde{M}, d_\phi^{(c)}), \quad (47)$$

in which  $\mathcal{G}_{\text{dil}}$  and  $d_\phi^{(c)}$  are defined respectively in (12) and (6) and the RG-time  $t$  in (3). This step is still compatible with the following dimensionless form of (40, 41):

$$S_{\text{int}} [\tilde{\phi}_1, t] = \frac{1}{2} (\tilde{M}_1 - \tilde{\phi}_1) \cdot \tilde{R} \cdot (\tilde{M}_1 - \tilde{\phi}_1) + \Gamma_{\text{int}} [\tilde{M}_1, t], \quad (48)$$

$$\tilde{M}_1 = \tilde{\phi}_1 - \tilde{R}^{-1} \cdot \frac{\delta}{\delta \tilde{\phi}_1} S_{\text{int}} [\tilde{\phi}_1, t], \quad (49)$$

in the sense that, with the additional condition  $\dot{S}_{\text{int}} = \dot{\Gamma}_{\text{int}}$ , these relations enable us to recover (23) from (47) –up to an additive constant– provided that  $\tilde{R} = -(\tilde{q}^2)^{n_0} / (K(\tilde{q}^2) - \text{const})$  which is compatible with the dimensionless version of (37) and with the limit of overall cutoff  $\Lambda_0 \rightarrow \infty$ .

Describing the way RG-step 3 has been commonly implemented is a bit more delicate.

#### 2.4.4 Field-renormalization

In principle, the field-renormalization step should be implemented purely “linearly” as done in section 2.3.3. This would correspond to an infinitesimal realization of RG-step 3 in response to the infinitesimal realization of RG-step 1. In practice it would consist in changing  $d_\phi^{(c)}$  by  $d_\phi^{(-)}$  [given by (8)] within

$\mathcal{G}_{\text{dil}}(\Gamma_{\text{int}}, \tilde{M}, d_\phi^{(c)})$  appearing in (47). It is easy to verify that, applying the procedure of the preceding section to (47) so modified, would not enable us to recover (27) or (29) without modifying the Legendre transformation<sup>15</sup>. This is because no explicit scale-dependence has been assumed within the cutoff function of the quadratic term in (40) that could compensate the field-renormalization effect.

It is a matter of fact that, instead of implementing it linearly (i.e. infinitesimally), the current introduction of  $\eta(t)$  within the flow equation of the effective action is based on a cutoff function involving a factorized scale-dependence that, precisely, compensates a global field-renormalization conveniently added to the infinitesimal implementation of the RG-steps. In the literature, we distinguish essentially two (very close) ways to proceed, one is due to Morris [39] and the other to Wetterich (for a review, see [16]).

**2.4.4.1 The Morris procedure** It is based on an “anomalous” dimensional analysis that relies upon the fact that, at a fixed point or very close to it, the effective dimension of the field is  $d_\phi^{*(-)}$  [see (8) with  $\eta(t) = \eta^*$ ]. As a consequence, to compensate this anomalous dimension introduced by hand, the cutoff function  $R$  (thus after the limit  $\Lambda_0 \rightarrow \infty$  has been performed) must display an anomalous dependence in the scale  $\Lambda$  such as (in [39],  $R$  is noted  $C^{-1}$ ):

$$R(q^2; \Lambda) = \Lambda^{2-\eta^*} \tilde{R}(\tilde{q}^2) . \quad (50)$$

The derivative w.r.t.  $\Lambda$ , occuring in the right hand side of (45), induces within the flow equation a nonlinear (w.r.t.  $\Gamma_{\text{int}}$ ) contribution proportionnal to  $\eta^*$  in addition to the usual linear contribution within  $\mathcal{G}_{\text{dil}}(\Gamma_{\text{int}}, \tilde{M}, d_\phi^{*(-)})$ . The ERGE for  $\Gamma_{\text{int}}$  reads thus [39]:

$$\dot{\Gamma}_{\text{int}}[\tilde{M}_1, t] = \frac{1}{2} \int_{\tilde{q}} \frac{[(\eta^* - 2) \tilde{R}(\tilde{q}^2) + 2\tilde{q}^2 \tilde{R}'(\tilde{q}^2)]}{\tilde{R}(\tilde{q}^2) + \Gamma_{\text{int}}^{(2)}[\tilde{q}; \tilde{M}_1]} + \mathcal{G}_{\text{dil}}(\Gamma_{\text{int}}, \tilde{M}, d_\phi^{*(-)}) . \quad (51)$$

**2.4.4.2 The Wetterich procedure** The cutoff function is assumed to display a global scale-dependent factor noted  $Z_\Lambda(e^{-t})$ , of the form:

$$R(q^2; \Lambda) = \Lambda^{2n_0} Z_\Lambda(e^{-t}) \tilde{R}(\tilde{q}^2) , \quad (52)$$

---

<sup>15</sup> For the same reason as one has “modified” the Polchinski equation (see section 2.3.3).

with [see eq.(10) in which  $Z_3 = 1/Z_\Lambda$ ]<sup>16</sup>:

$$\eta(t) - 2(1 - n_0) = -\Lambda \frac{\partial}{\partial \Lambda} Z_\Lambda(e^{-t}) = \frac{d}{dt} Z_\Lambda(e^{-t}), \quad (53)$$

that, de facto introduces arbitrarily –within the cutoff function– an effect of a preliminar<sup>17</sup> decimation over the finite range  $[\Lambda, \mu_0]$ , see (3). Then Wetterich defines a “dimensionless-renormalized” field  $\tilde{M}_1$  as  $M_1 = [Z_\Lambda(t)]^{-1/2} \Lambda^{d/2-n_0} \tilde{M}_1$  –corresponding to a field-renormalization over the finite range  $[\Lambda, \mu_0]$  that compensates the previous effect (or perhaps it is the reverse). It is easy to verify that, so doing, one obtains the same equation as Morris (51) with a significant difference, however, that, this time,  $\eta(t)$  is not necessarily fixed to  $\eta^*$ , but is allowed to flow (its flow is, in principle, dictated by the constancy of the coefficient of the kinetic term of  $\Gamma$  though it is not necessarily determined that way in [16]).

**2.4.4.3 Back to the full action** Because the explicit scale-dependence of the cutoff function now compensates exactly the field-renormalisation, one may utilize the Legendre transformation (48, 49) to get, from (51), the corresponding form of the RG-flow equation for  $S_{\text{int}}$  and then for<sup>18</sup>  $S$ . Obviously one does not recover (27) or (29) that way. Nevertheless one obtains a RG-flow equation for  $S$  with a non-linear dependence on  $\eta(t)$  that may be directly compared with the historic Wilson equation (or, rather, its version “*extended to an arbitrary cutoff function*”). It appears that the non-linear contribution proportional to  $\eta(t)$  found that way is opposite in sign compared to that of the historic first version (for more detail, see [18]). As we shall see, this difference in sign is due to the fact that the Morris-Wetterich procedure of introducing  $\eta(t)$ , though perfectly acceptable, is artificial (forced) and not directly associated to the realization of RG-step 1.

Is, this difference in sign, the reason why one has rather tried to adapt the Legendre transformation to link the *linear* version of Ball et al to the *nonlinear* version of Morris or Wetterich [6–8]? It is hard to answer that question but we may a priori expect a complicated form of the Legendre transformation in that case.

<sup>16</sup>  $Z_\Lambda(e^{-t})$  is the inverse of the usual wave function renormalization  $Z_3(e^{-t})$  which relates the “bare” field  $\phi$  to the renormalized field  $\phi_R$  as  $\phi = \sqrt{Z_3(e^{-t})} \phi_R$ . Also, in [16], Wetterich’s RG-time is the opposite of (3),  $n_0 = 1$ , and  $k$  and  $\Lambda$  correspond respectively to our  $\Lambda$ ,  $\mu_0$ .

<sup>17</sup> Prior to the infinitesimal change of  $\Lambda$  of interest to derive the RG equation.

<sup>18</sup> Having the flow for  $S_{\text{int}}$  one may use the relation (24) back to  $S$ . This does not mean, however, that a Legendre transformation links directly the flow equation satisfied by  $S$  to that satisfied by  $\Gamma_{\text{int}}$ .

### 2.4.5 Elaborated Legendre transformation

Studies [6–8] have shown that one can define a Legendre transformation that relates the “modified” Polchinski flow equation on the one hand to the flow equation (51) satisfied by  $\Gamma_{\text{int}}$  (even in the case of  $\eta(t)$  not limited to be a constant [7]) on the other hand. Of course, because  $\eta(t)$  is, notably, not introduced in the same way in the two flow equations, this Legendre transformation is extremely complicated (but this is not the only reason). It is obtained using a generalization of (40, 41) written in terms of dimensionless quantities as follows:

$$S_{\text{int}}[\tilde{\phi}] = \Gamma_{\text{int}}[\tilde{M}] + \frac{1}{2}\tilde{M} \cdot \mathcal{R} \cdot \tilde{M} + \frac{1}{2}\tilde{\phi} \cdot \mathcal{Q} \cdot \tilde{\phi} - \tilde{\phi} \cdot \mathcal{L} \cdot \tilde{M},$$

$$\mathcal{L} \cdot \tilde{M} = \mathcal{Q} \cdot \tilde{\phi} - \frac{\delta}{\delta \tilde{\phi}} S_{\text{int}}[\tilde{\phi}],$$

in which the functions  $\mathcal{R}(\tilde{q}^2)$ ,  $\mathcal{Q}(\tilde{q}^2)$  and  $\mathcal{L}(\tilde{q}^2)$  are adjusted so as to get (51) from (29). With a choice of cutoff function corresponding to (19) and  $n_0 = 1$ , the solution reads [6]:

$$\mathcal{Q}(x) = \frac{x}{K(x)} \left( \frac{1}{\sigma(x)} - 1 \right),$$

$$\mathcal{L}(x) = \frac{x}{\sigma(x)},$$

$$\mathcal{R}(x) = x \frac{K(x)}{\sigma(x)},$$

with:

$$\sigma(x) = K(x) x^{\eta^*/2} \int_0^x du u^{-\eta^*/2} \frac{d}{du} \frac{1}{K(u)}.$$

Of course, in the scale-dependent case  $\eta(t)$ , the expressions are more involved [7].

## 3 Structural method

### 3.1 Reminder of the “analytical method”

Except for the historic first ERGE [1], the common way of obtaining the RG-flow equations for  $S_\Lambda$  is based on considering first  $S_{\text{int},\Lambda}$  and an explicit

UV-cutoff function. Let us , call “analytical method” the procedure of infinitesimally performing the three RG-steps first on  $S_{\text{int},\Lambda}$ . A normal analytic method may be sketched as follows (see part 2):

- (1) RG-step 1 is performed on  $S_{\text{int},\Lambda}$  through an infinitesimal variation of the explicit UV-cutoff as introduced in (18),
- (2) the relation between  $S_\Lambda$  and  $S_{\text{int},\Lambda}$  (18) is then considered to transfer the preceding result on  $S_\Lambda$ ,
- (3) the rescaling RG-step 2 is (infinitesimally) performed both on  $S$  and  $S_{\text{int}}$  to give the RG-flow equations for  $S[\tilde{\phi}, t]$  and  $S_{\text{int}}[\tilde{\phi}, t]$  [this commutes with the preceding step since the relation (18) becomes (24) which has the same form and induces the equality (25)  $\dot{S}[\tilde{\phi}, t] = \dot{S}_{\text{int}}[\tilde{\phi}, t]$ ],
- (4) the field-renormalization RG-step 3 is implemented linearly (i.e. infinitesimally) within the RG-flow equation for either  $S_{\text{int}}[\tilde{\phi}, t]$  or  $S[\tilde{\phi}, t]$ ,
- (5) the final relation that enables to deduce one RG-flow equation from the other is altered compared to (25), then one must modify either the relation (25) or one of the two RG-equations so obtained (hence the so-called “modified” Polchinski equation).

A similar strategy could be sketched that characterizes a normal way of treating the Legendre transformation linking the RG-flow equations for  $S_{\text{int},\Lambda}$  and  $\Gamma_{\text{int},\Lambda}$ .

As indicated in sections 2.2 and 2.4.4, Wilson on the one hand and Morris-Wetterich on the other hand have not followed the “normal” way concerning the realization of RG-step 3 since they have both used a non-infinitesimal field renormalization. The Legendre transformation that links the Morris-Wetterich RG-flow equation to the “normal” Polchinski flow equation is very complicated.

Instead of adapting a Legendre transformation so as to obtain the Morris-Wetterich RG-flow equation (51) –as done in [6–8], we aim at determining the flow equation satisfied by  $\Gamma_{\text{int}}$  that corresponds to a Wilsonian ERGE “*extended to an arbitrary cutoff function*”, while maintaining a simple form for the Legendre transformation and for the link between  $S$  and  $S_{\text{int}}$ . Because the method followed relies on structural properties of the ERGE, we call it the structural method.

### 3.2 Principle of the structural method

The basical idea is to implement RG-step 3 via an EMRO of the flow equation for  $S[\tilde{\phi}, t]$ . Indeed, an EMRO is an “operator” responsible for the infinitesimal change of field-normalization by a constant that moves the action

infinitesimally along a line of equivalent fixed points. Then, having extended its expression out of the vicinity of fixed points, one may use it to implement the field-renormalization step in the ERGE. This looks like the Wilson procedure though the reasoning is inversed. In the historic first version, the implementation of RG-step 3 is done via the operator that actually infinitesimally changes the normalization of the field which, at a fixed point and for the Wilson realization of the RG-step 1, coincides with an EMRO (this is not always true, e.g. see [18] and appendix B). Of course we could continue to introduce RG-step 3 the same way as Wilson did [i.e. using (13)] whatever the realization of RG-step 1, but then a Legendre transformation as simple as that constructed in this article might not be so easily obtained.

At first sight, it seems that it could be impossible to determine an EMRO attached to a Wilsonian RG-flow equation which is incompletely known. Fortunately, the operation of changing the normalization of the field by a constant commutes with the RG-steps [36]. Then one may determine an EMRO of a given RG-flow equation without having implemented RG-step 3 yet. From this knowledge we may complete the RG-steps via the EMRO so determined, being reassured that it will remain EMRO of the modified flow equation [18] (see also appendix B). From the general form (15) of a redundant operator, it is possible to determine a strategy to find the expression of an EMRO attached to a Wilsonian RG-flow equation [15] (see appendix B).

By extension of its definition, let us call also EMRO a solution of an eigenvalue-RG-equation<sup>19</sup> with a zero eigenvalue whatever the RG equation considered. Due to its universal character, the whole spectrum of eigenvalues does not depend on the kind of ERGE considered. For a scalar field and a given dimension  $d$ , the spectrum is the same whether one considers the Wilson, the Polchinski or the Wetterich ERGE. So, each kind of equation possesses an EMRO (in its extended meaning). Then if one knows the expression of an EMRO for a given ERGE, one may deduce the expression of the corresponding EMRO for any other ERGE provided that relations between the various flow equations are given. The structural method thus consists in finding the expressions of the various ERGE that correspond to the Wilson-like ERGE within which RG-step 3 has been implemented via an EMRO and for given simple (but justified) relations between the various actions.

### 3.3 The Wilson ERGE “extended to an arbitrary cutoff function”

The first step is to establish the expression of the Wilson ERGE –with the field-renormalization implemented via an EMRO– “extended to an arbitrary cutoff function” [18, 42].

---

<sup>19</sup> Linearization of the RG flow in the vicinity of a fixed point.

### 3.3.1 Structural derivation

It is possible to deduce it, structurally, from the considerations of sections 2.2 and 2.3.

Considering a cutoff function  $P(q^2, \Lambda)$  as introduced in (18), one may infer the following flow equation for  $S[\tilde{\phi}, t]$ , provided  $n_0 = 0$ :

$$\begin{aligned} \dot{S} = & - \int_{\tilde{q}} \tilde{q}^2 \tilde{P}'(\tilde{q}^2) \left[ \frac{\delta^2 S}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} + 2 \tilde{P}^{-1}(\tilde{q}^2) \tilde{\phi}_{\tilde{q}} \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \right] \\ & + \mathcal{G}_{\text{dil}}(S, \tilde{\phi}, d_{\phi}^{(cw)}) + \varpi_0(t) \mathcal{O}_P(S, \tilde{\phi}) , \end{aligned} \quad (54)$$

in which  $d_{\phi}^{(cw)}$  is given by (7) and  $\mathcal{O}_P(S^*, \tilde{\phi})$  is an EMRO associated to the flow equation linearized about a fixed point  $S^*$ . Written under the general form (15),  $\mathcal{O}_P(S, \tilde{\phi})$  (considered out of the fixed point) corresponds to:

$$\psi_{\tilde{q}}^{(P)}(\tilde{\phi}, S) = \tilde{\phi}_{\tilde{q}} - \tilde{P}(\tilde{q}^2) \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} , \quad (55)$$

that is to say:

$$\mathcal{O}_P(S, \tilde{\phi}) = \int_{\tilde{q}} \left[ \tilde{P}(\tilde{q}^2) \left( \frac{\delta^2 S}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S}{\delta \tilde{\phi}_{-\tilde{q}}} \right) + \tilde{\phi}_{\tilde{q}} \frac{\delta S}{\delta \tilde{\phi}_{\tilde{q}}} - 1 \right] , \quad (56)$$

see [18] and appendix B for more detail.

Indeed, (54) is the flow equation for  $S$  as obtained by Ball et al [14], using the analytical method sketched in section 3.1, but in which the field-renormalization step is implemented via  $\mathcal{O}_P(S, \tilde{\phi})$  instead of being implemented linearly within  $\mathcal{G}_{\text{dil}}$  exclusively. Moreover,  $\mathcal{O}_P(S, \tilde{\phi})$  may be deduced from  $\mathcal{O}_w(S, \tilde{\phi})$  (13) after the change<sup>20</sup>  $\tilde{\phi} \rightarrow \tilde{\phi}/\sqrt{\tilde{P}}$ .

**Remark 5** *In order to preserve the quasi-local character of the action after the field-redefinition that allows to recover the Wilson EMRO (13), it is necessary that  $n_0 = 0$ . Consequently, in the following, if not mentioned, we assume that  $n_0 = 0$ , i.e.  $d_{\phi}^{(c)} \rightarrow d_{\phi}^{(cw)}$ . (Actually the EMRO for  $n_0 = 1$  takes on a simple form also though it is a bit modified compared to (56) in order to preserve quasi-locality, see appendix B.)*

---

<sup>20</sup> Though we could have expected that the EMRO be still the infinitesimal realization of the operator  $U_{\kappa}$ , as given by (14), which is responsible for a change of normalization of the field  $\tilde{\phi}$ .



If the above derivation of (54) is satisfactory, it is complicated due to the necessity of determining first the expression of an EMRO. Indeed we have been able to find (56) because of the preliminary knowledge of the original Wilson’s EMRO (13) which itself was known to be constructed from the explicit transformation (14). However, because of the numerous possibilities of implementing RG-step 1, and the freedom of redefining the field at will (the fourth step of the RG-procedure), an EMRO is not always so directly related to (13) (see appendix B). Moreover, the role of the cutoff function, seen as a field redefinition, is partly hidden in this approach so that it is instructive to look also at the “analytical” way of obtaining (54) based on the consideration of an explicit UV-cutoff function. As already encountered in section 2.4.4, the non-linear introduction of  $\eta(t)$  may be realized by assuming that the cutoff function displays a factorized scale-dependence. In the following section we look at this analytical way of obtaining (54) and “explain” how and why it produces a contribution opposite in sign compared to the Morris-Wetterich way of realizing RG-step 3. We will also observe that the recourse to an EMRO implies, de facto, that the usual limit of infinite overall cutoff ( $\Lambda_0 \rightarrow \infty$ ) be effected which finally justifies the simplicity of the Legendre transformation (1, 2) to be used in the structural approach.

### 3.3.2 Analytical derivation

In [42] a derivation of (54) was given based on an explicit ad-hoc scale-dependence within the cutoff function introduced by the following change:

$$\tilde{P}(\tilde{q}^2) \rightarrow \zeta(t) \tilde{P}(\tilde{q}^2) , \quad (57)$$

$$\zeta(t) = e^{-(2-\eta^*)t} . \quad (58)$$

As already mentioned at the end of section 2.4.4 and emphasized in [18], the sign of the exponent in (58) is opposite to that assumed by Morris and Wetterich within the cutoff function  $R$  [see eqs (50), (52, 53) with  $n_0 = 0$  and  $R \sim 1/\tilde{P}$ ]. An explanation of that latter point and a clear justification of (57, 58) was not given<sup>21</sup> in [42]. It is thus justified to present here a detailed re-derivation and justification of (54) on the basis of a cutoff function involving a factorized scale-dependence.

First, let us come back to the discussion of section 2.4 concerning the obtention of Polchinski’s equation (35) from an IR cutoff function and its relation to the

---

<sup>21</sup> In [42], not only the origin of the change (57, 58) was not strongly justified but also some misleading arguments have been given. In particular a confusion has been made between  $\zeta(t)$  and the inverse of  $Z_3(e^{-t})$ . The reason for this confusion is “explained” below.

original procedure via (36, 37). In original Polchinski's paper [13], the running cutoff  $\Lambda$  is like an overall cutoff  $\Lambda_0$ , whereas the running cutoff  $\Lambda$  of section 2.4 is the result of having integrated the degrees of freedom attached to the finite momentum range  $[\Lambda, \Lambda_0]$ . Thus, for the scale-dependent action  $S_\Lambda[\phi_1]$  to be like the Polchinski action, two steps have still to be carried out:

- (1) the rescaling  $\Lambda \rightarrow \Lambda_0$
- (2) the field-renormalization  $\phi_1 \rightarrow \sqrt{Z_3\left(\frac{\mu}{\Lambda}\right)}\phi_R$  with  $\mu < \Lambda$ .

Before going any further, let us draw attention to the following subtleties:

- a. in the system of units set by the overall cutoff  $\Lambda_0$ ,  $\mu$  represents the momentum-scale value reached by the running scale  $\Lambda$  *after* decimation and *before* rescaling. After rescaling, the running scale  $\Lambda$  takes on the initial value  $\Lambda_0$  whereas the overall cutoff  $\Lambda_0$  is increased at the same rate. Obviously, the ratio  $\mu/\Lambda$  has the same magnitude as the ratio  $\Lambda/\Lambda_0$ , but the field-renormalization function  $Z_3$  depends on the ratio  $\mu/\Lambda$  (corresponding to the range over which the decimation is effected) and not on the ratio  $\Lambda/\Lambda_0$  (contrary to Wetterich's procedure of field-renormalization<sup>22</sup>, see section 2.4.4). Thus, the dependence on the running scale  $\Lambda$  within  $Z_3\left(\frac{\mu}{\Lambda}\right)$  occurs rightly, but unusually [e.g., in comparison to (53)], in the denominator of the scale ratio so that, according to (10),  $\Lambda \frac{d}{d\Lambda} Z_3\left(\frac{\mu}{\Lambda}\right) = -\ell \frac{d}{d\ell} Z_3(\ell) = \frac{d}{dt} Z_3(e^{-t}) = 2\varpi_0(t) Z_3$ .
- b. the usual limit of infinite initial cutoff  $\Lambda_0 \rightarrow \infty$  at fixed  $\Lambda$  induces  $\mu \rightarrow 0$ . That is to say, taking that limit would be equivalent to assuming that, in some sense, the RG program would have already been entirely (pre)-effected over the whole momentum range  $[0, \infty[$  whereas  $\Lambda$  would remain finite and non zero! We will see that this apparently strange procedure concerns only the field-renormalization and is obliged in order to "analytically explain" the implementation of RG-step 3 via an EMRO.

For now we still assume that  $\Lambda_0$  is finite. Then, the completion of the field-renormalization step over the finite range  $[\mu, \Lambda]$  implies that (37) becomes:

$$P(q^2, \Lambda) = \left[ Z_3\left(\frac{\mu}{\Lambda}\right) \right]^{-1} \Delta_1(q^2, \Lambda). \quad (59)$$

If, following Polchinski, one considers that  $P(q^2, \Lambda)$  corresponds to the true origin of the RG-time, then  $\Lambda = \Lambda_0$  and thus  $\frac{\mu}{\Lambda} = 1$  that is to say  $Z_3 = 1$ , consequently (59) coincides with (37). Notice that considering the limit

---

<sup>22</sup> Although, in Wetterich's procedure,  $\Lambda_0$  has already been sent to infinity and is implicitly replaced by an arbitrary momentum-scale  $\mu_0$ , the argument is not altered.

$\Lambda_0 \rightarrow \infty$  has no meaning in the circumstances. That is the genuine Polchinski starting point.

Instead, but equivalently, one can imagine that it exists an overall cutoff  $\Lambda_0 > \Lambda$  consequence of the implementation of a complete RG procedure (three steps) over the finite range  $[\Lambda, \Lambda_0]$  and preliminary to the infinitesimal reduction of  $\Lambda$ . One may still consider that, nevertheless,  $P(q^2, \Lambda)$  is unchanged compared to the original Polchinski version (after the RG procedure the system is unchanged except for a reduction of the momentum range). In that case the relation (59) would differ from (37) –because of the decimation that would have been actually effected over the range  $[\mu, \Lambda]$  with  $\mu < \Lambda$ . Indeed, to keep  $P(q^2, \Lambda)$  unchanged after the field renormalization over the range  $[\mu, \Lambda]$ ,  $\Delta_1(q^2, \Lambda)$  must have acquired –prior to field renormalization– a factorized scale-dependence<sup>23</sup> given by the inversion of (59):

$$\Delta_1(q^2, \Lambda) = \left[ Z_3 \left( \frac{\mu}{\Lambda} \right) \right] P(q^2, \Lambda) . \quad (60)$$

Clearly, in terms of  $\phi_R$ , Polchinski’s procedure would remain unchanged and his equation would not be modified. Nevertheless, this time, a totally ineffective finite overall cutoff  $\Lambda_0$  would (implicitly) exist –together with a possible implicit IR-momentum-scale  $\mu$ , see below. However, these latter two scales do not alter the RG-flow because only the local (infinitesimal) variation of  $\Lambda$  is of interest.

From this, we may foresee a new point of view. Let us suppose that instead of effecting the three RG-steps over the finite range  $[\Lambda, \Lambda_0]$  *before* considering the construction of the Polchinski ERGE (via an infinitesimal variation of  $\Lambda$ ), we assume that only RG-step 1 and 2 have been effected. Thus, in order to complete the three RG-steps within the ERGE, the field-renormalization will have still to be effected globally over the finite range  $[\mu, \Lambda]$  instead of being implemented only infinitesimally. Then, to start with the derivation of the ERGE (via an infinitesimal variation of  $\Lambda$ ), let us suppose that the field of reference is the “unrenormalized” field  $\phi_1$  (the “old field” would say Bell and Wilson [36]). Consequently, Eq. (60) defines  $\Delta_1(q^2, \Lambda)$  in terms of the regular UV cutoff function  $P(q^2, \Lambda)$ . In that case, (33) reads:

$$\Delta_2(q^2; \Lambda, \Lambda_0) = \Delta_0(q^2, \Lambda_0) - \left[ Z_3 \left( \frac{\mu}{\Lambda} \right) \right] P(q^2, \Lambda) , \quad (61)$$

so that the derivative w.r.t.  $\Lambda$  in the Polchinski equation (35) generates a contribution proportional to  $\left[ \Lambda \frac{d}{d\Lambda} Z_3 \left( \frac{\mu}{\Lambda} \right) \right] = 2\varpi_0(t) Z_3 \left( \frac{\mu}{\Lambda} \right)$  which has a sign

---

<sup>23</sup> Which may be seen as consequence of the decimation –as it stands for any coefficient of the scale-dependent action  $S[\tilde{\phi}, t]$  prior to field-renormalization.

opposite to what would be obtained with the Morris-Wetterich procedures described in section 2.4.4. We emphasize again that this change of sign is not due to Wetterich's choice of  $Z_\Lambda = 1/Z_3$  but rather to the fact that  $\Lambda$  occurs *unusually, but correctly*, in the denominator of the scale ratio within  $Z_3$ .

The procedure for obtaining the ERGE is unchanged relatively to the first two steps: an infinitesimal change  $\Lambda \rightarrow \Lambda - d\Lambda$  with its associated infinitesimal rescaling (with a classical  $\mathcal{G}_{\text{dil}}$ ); however the third step (field-renormalization) has to be realized over the full finite range  $[\mu, \Lambda]$  instead of being implemented infinitesimally and this induces a subtlety in the procedure. Actually the factor  $\left[Z_3\left(\frac{\mu}{\Lambda}\right)\right]$  in (61) has anticipated the *consequence* of the infinitesimal realization of the RG-step 1 over the range  $[\Lambda - d\Lambda, \Lambda] \subset [\mu, \Lambda]$  which is under present consideration. Hence, this part of  $\left[Z_3\left(\frac{\mu}{\Lambda}\right)\right]$  must be taken away from the derivative w.r.t.  $\Lambda$ . So, to avoid an overlapping between the  $\Lambda$ -dependencies in  $Z_3$  and  $P$  (i.e., to avoid a possible double counting), we must effectuate a “pre-renormalization” of the field ( $\phi \rightarrow \sqrt{Z_3\left(\frac{\Lambda-d\Lambda}{\Lambda}\right)}\phi$ ) over the infinitesimal range<sup>24</sup> since then the infinitesimal RG-step 1 will regenerate the full factor  $Z_3\left(\frac{\mu}{\Lambda}\right)$  in front of  $P$ . After that, the final field-renormalization can be truly performed over the full range  $[\mu, \Lambda]$ . Finally, after completing the field renormalization, but not yet the infinitesimal rescaling, we get:

$$\begin{aligned} \Lambda \frac{\partial}{\partial \Lambda} S_{\text{int}}[\phi]_\Lambda \Big|_\phi &= -\frac{1}{2} \int_q \left\{ \left[ 2\varpi_0(t) P(q^2; \Lambda) + \Lambda \frac{\partial P(q^2; \Lambda)}{\partial \Lambda} \right] \Big|_q \right. \\ &\quad \times \left[ \frac{\delta^2 S_{\text{int}}[\phi]_\Lambda}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S_{\text{int}}[\phi]_\Lambda}{\delta \phi_q} \frac{\delta S_{\text{int}}[\phi]_\Lambda}{\delta \phi_{-q}} \right] \Big\} \\ &\quad + \varpi_0(t) \phi \cdot \frac{\delta}{\delta \phi} S_{\text{int}}[\phi]_\Lambda, \end{aligned} \quad (62)$$

in which  $\phi$  stands for  $\phi_R$  and the last term is due to the required “pre-renormalization” with  $\varpi_0(t)$  given by (10).

After rescaling we obtain:

$$\begin{aligned} \dot{S}_{\text{int}}[\tilde{\phi}, t] &= \int_{\tilde{q}} \left( \varpi_0(t) \tilde{P}(\tilde{q}^2) - \tilde{q}^2 \tilde{P}'(\tilde{q}^2) \right) \left[ \frac{\delta^2 S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{-\tilde{q}}} \right] \\ &\quad + \mathcal{G}_{\text{dil}}(S_{\text{int}}, \tilde{\phi}, d_\phi^{(+)}(t)), \end{aligned} \quad (63)$$

in which  $d_\phi^{(+)}(t)$ , defined in (8), differs from the usual  $d_\phi^{(-)}(t)$ , but is quite warranted.

---

<sup>24</sup> As we did it above, over the whole range  $[\mu, \Lambda]$ , to recover Polchinski's procedure.

Then it is easy to verify that the Wilsonian RG-flow (54) –under its exact writing– may be obtained either from (62) and the usual relation (18) –with the rescaling step finally completed on  $S$ – or directly from (63) and the dimensionless form (24) –with the condition (25).

### 3.3.3 Additional important comment

Though neither  $\mu$  nor  $\Lambda_0$  appear explicitly in the RG-flow equations for  $S$  and  $S_{\text{int}}$  obtained in the preceding section, the analytical method followed relies upon an explicit reference to them (via  $\mu$  occurring within the cutoff function). Contrary to  $\Lambda_0$  that obviously has no effect on the usual Polchinski RG-flow equation, the reference to  $\mu$  is not completely harmless. Remind that  $\mu \neq 0$  is a consequence of implementing the field-renormalization corresponding to the decimation over the range  $[\Lambda, \Lambda_0]$  (i.e. those fields having momentum components in the finite range  $[\mu, \Lambda]$  with  $\mu < \Lambda$ ). Then, because the RG-flow corresponds to a decreasing of  $\Lambda$ , it is almost obvious that  $\mu$  is formally an unacceptable lower limit to the running scale  $\Lambda$ . This is because for  $\Lambda < \mu$  the field-renormalization is no longer accounted for within  $Z_3(\mu/\Lambda)$  and, thus, should be completed the usual linear way (what has not been expressed in the ERGE). Though  $\mu$  does not appear in our writing of the RG-flow equation, it should<sup>25</sup>. Thus, strictly speaking, the analytical method reveals that the RG-flow equation in which RG-step 3 is implemented via an EMRO either is only valid<sup>26</sup> for  $\Lambda > \mu$  or implies the limit  $\mu \rightarrow 0$  be effected. Now  $\mu$  plays the role of an overall IR-cutoff that is the counter-part of the UV-cutoff  $\Lambda_0$ . It is almost obvious that, to make a complete contact with the Wilson procedure of implementing RG-step 3 using an EMRO at any scale, the limit  $\mu \rightarrow 0$  must be explicitly achieved (together with the concomitant limit  $\Lambda_0 \rightarrow \infty$ ). This point, that may be connected to remark 2, will be made even clearer with the consideration of the RG-flow equation for the effective action  $\Gamma$ , see sections 3.4.2 and 3.4.3.

## 3.4 Structural constructions

Because the simple Legendre transformation (1,2) is not yet justify, we still need to use partly the analytical version requiring to consider first  $S_{\text{int}}$  and  $\Gamma_{\text{int}}$  instead of considering exclusively to the full actions  $S$  and  $\Gamma$ . Thus, having

---

<sup>25</sup> For the same reason we should mention that the ERGE is only valid for  $\Lambda < \Lambda_0$ , what is never done with a Polchinski like equation in which  $\Lambda_0$  never appears explicitly.

<sup>26</sup> This limitation does not occur in the Morris-Wetterich way of implementing RG-step 3 because it is based on the introduction of the ad-hoc factor  $Z_3(\Lambda/\mu_0)$  in front of the cutoff function with  $\mu_0 > \Lambda$ .

justified both structurally and analytically<sup>27</sup> the form of (54) we may lean on it to rederive structurally the Polchinski-like ERGE (63) and, via the usual Legendre transform, the corresponding Wetterich-like ERGE.

### 3.4.1 Polchinski-like ERGE

After implementation of the three RG-steps, and owing to the factorized scale-dependence of the cutoff function (60) that entirely absorbs the global field-renormalization step over the range  $[\mu, \Lambda]$ , we may assume the relations (24, 25) between  $S[\tilde{\phi}, t]$  and  $S_{\text{int}}[\tilde{\phi}, t]$ . Then, considering the expression (56) for the EMRO of the extended Wilson ERGE (54), we may easily find its expression in terms of  $S_{\text{int}}$  using (24). The resulting expression should be the expression of the EMRO associated to the Polchinski ERGE, it comes [18]:

$$\mathcal{O}_P(S_{\text{int}}, \tilde{\phi}) = \int_{\tilde{q}} \left[ \tilde{P}(\tilde{q}^2) \left( \frac{\delta^2 S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{-\tilde{q}}} \right) - \tilde{\phi}_{\tilde{q}} \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}}} \right], \quad (64)$$

Finally, from the general property:

$$\mathcal{G}_{\text{dil}} \left( \frac{1}{2} \tilde{\phi} \cdot \tilde{P}^{-1} \cdot \tilde{\phi}, \tilde{\phi}, d_{\phi}^{(c)} \right) = \int_{\tilde{q}} \tilde{\phi}_{\tilde{q}} \left[ n_0 \tilde{P}^{-1} - \tilde{q}^2 (\tilde{P}^{-1})' \right] \tilde{\phi}_{-\tilde{q}}, \quad (65)$$

in which we actually set  $n_0 = 0$ , it is easy to verify that the ERGE satisfied by  $S_{\text{int}}$  that accounts for the three RG-steps structurally deduced from (54) reads [18]:

$$\begin{aligned} \dot{S}_{\text{int}} = & - \int_{\tilde{q}} \tilde{q}^2 \tilde{P}'(\tilde{q}^2) \left[ \frac{\delta^2 S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}} \delta \tilde{\phi}_{-\tilde{q}}} - \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}}} \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{-\tilde{q}}} \right] \\ & + \mathcal{G}_{\text{dil}}(S_{\text{int}}, \tilde{\phi}, d_{\phi}^{(cw)}) + \varpi_0(t) \mathcal{O}_P(S_{\text{int}}, \tilde{\phi}). \end{aligned} \quad (66)$$

If one splits the contribution proportionnal to  $\mathcal{O}_P(S_{\text{int}}, \tilde{\phi})$  in two parts, we then recover (63) as it must.

It is also interesting to express the general form of a redundant operator (15) in terms of  $S_{\text{int}}$ , it comes:

$$\mathcal{O}(S_{\text{int}}, \tilde{\phi}) = \int_{\tilde{q}} \left[ \bar{\psi}_{\tilde{q}}(\tilde{\phi}, S_{\text{int}}) \left( \tilde{P}^{-1}(\tilde{q}^2) \tilde{\phi}_{-\tilde{q}} + \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{\tilde{q}}} \right) - \frac{\delta \bar{\psi}_{\tilde{q}}(\tilde{\phi}, S_{\text{int}})}{\delta \tilde{\phi}_{\tilde{q}}} \right]. \quad (67)$$

---

<sup>27</sup> At least for  $\mu < \Lambda$ , see section 3.3.3.

Thus (64) corresponds to:

$$\bar{\psi}_{\tilde{q}}^{(P)}(\tilde{\phi}, S_{\text{int}}) = -\tilde{P}(\tilde{q}^2) \frac{\delta S_{\text{int}}}{\delta \tilde{\phi}_{-\tilde{q}}} . \quad (68)$$

### 3.4.2 Wetterich-like ERGE

Similarly to the preceding section, we may deduce from (64) the expression of the EMRO in terms of  $\Gamma_{\text{int}}$ . To this end we use the Legendre transformation (40, 41) rewritten in terms of renormalized dimensionless quantities (i.e. after rescaling and field-renormalization) that reads:

$$S_{\text{int}}[\tilde{\phi}, t] = \frac{1}{2} (\tilde{M} - \tilde{\phi}) \cdot \bar{\Delta}_2^{-1} \cdot (\tilde{M} - \tilde{\phi}) + \Gamma_{\text{int}}[\tilde{M}, t] , \quad (69)$$

$$\tilde{M} = \tilde{\phi} - \bar{\Delta}_2 \cdot \frac{\delta}{\delta \tilde{\phi}} S_{\text{int}}[\tilde{\phi}, t] , \quad (70)$$

$$\dot{S}_{\text{int}}[\tilde{\phi}, t] = \dot{\Gamma}_{\text{int}}[\tilde{M}, t] , \quad (71)$$

in which  $\tilde{M}$  is a “dimensionless renormalized” field defined similarly to  $\tilde{\phi}$  and

$$\bar{\Delta}_2(\tilde{q}^2, t) = \left[ Z_3 \left( \frac{\mu}{\Lambda} \right) \right]^{-1} \tilde{\Delta}_2 \left( \tilde{q}^2, t, \frac{\Lambda_0}{\mu_0} \right) ,$$

is the dimensionless IR-cutoff function after field-renormalization. Using (33) and (61), it reads also:

$$\bar{\Delta}_2(\tilde{q}^2, t) = \left[ Z_3 \left( \frac{\mu}{\Lambda} \right) \right]^{-1} \tilde{\Delta}_0 \left( \tilde{q}^2, \frac{\Lambda}{\Lambda_0} \right) - \tilde{P}(\tilde{q}^2) . \quad (72)$$

The expression of the EMRO (64) thus leads to:

$$\begin{aligned} \mathcal{O}_P(\Gamma_{\text{int}}, \tilde{M}) = \int_{\tilde{q}} \left\{ \left[ \tilde{P}(\tilde{q}^2) \frac{\Gamma_{\text{int}}^{(2)}[\tilde{q}; \tilde{M}]}{1 + \bar{\Delta}_2 \Gamma_{\text{int}}^{(2)}[\tilde{q}; \tilde{M}]} \right. \right. \\ \left. \left. - \left( \tilde{P}(\tilde{q}^2) + \bar{\Delta}_2(\tilde{q}^2, t) \right) \frac{\delta \Gamma_{\text{int}}}{\delta \tilde{M}_{\tilde{q}}} \frac{\delta \Gamma_{\text{int}}}{\delta \tilde{M}_{-\tilde{q}}} \right] - \tilde{M}_{\tilde{q}} \frac{\delta \Gamma_{\text{int}}}{\delta \tilde{M}_{\tilde{q}}} \right\} \quad (73) \end{aligned}$$

and the flow equation for  $\Gamma_{\text{int}}$ , deduced from (63) or (66), reads:

$$\begin{aligned}
\dot{\Gamma}_{\text{int}} [\tilde{M}] = & - \int_q q^2 \tilde{P}'(q^2) \left\{ \frac{\Gamma_{\text{int}}^{(2)} [\tilde{q}; \tilde{M}]}{1 + \bar{\Delta}_2 \Gamma_{\text{int}}^{(2)} [\tilde{q}; \tilde{M}]} \right\} \\
& + \int_q \left\{ q^2 [\bar{\Delta}'_2(\tilde{q}^2, t) + \tilde{P}'(q^2)] \frac{\delta \Gamma_{\text{int}}}{\delta \tilde{M}_{-q}} \frac{\delta \Gamma_{\text{int}}}{\delta \tilde{M}_q} \right\} \\
& + \mathcal{G}_{\text{dil}}(\Gamma_{\text{int}}, \tilde{M}, d_\phi^{(cw)}) + \varpi_0(t) \mathcal{O}_P(\Gamma_{\text{int}}, \tilde{M}) ,
\end{aligned} \tag{74}$$

in which  $\bar{\Delta}'_2(\tilde{q}^2, t) = \partial \bar{\Delta}_2(\tilde{q}^2, t) / \partial \tilde{q}^2|_t$ . To get this results we have used the property that, for  $n_0 = 0$ :

$$\mathcal{G}_{\text{dil}}(S_{\text{int}}, \tilde{\phi}, d_\phi^{(cw)}) = \mathcal{G}_{\text{dil}}(\Gamma_{\text{int}}, \tilde{M}, d_\phi^{(cw)}) + \int_{\tilde{q}} \left\{ \tilde{q}^2 \bar{\Delta}'_2(\tilde{q}^2, t) \frac{\delta \Gamma}{\delta \tilde{M}_{-\tilde{q}}} \frac{\delta \Gamma}{\delta \tilde{M}_{\tilde{q}}} \right\} .$$

Similarly to the preceding section, it is interesting to express, for given  $\tilde{P}$  and  $\bar{\Delta}_2$  the general form of a redundant operator (15) in terms of  $\Gamma_{\text{int}}$ , it comes:

$$\begin{aligned}
\mathcal{O}(\Gamma_{\text{int}}, \tilde{M}) = & \int_{\tilde{q}} \left\{ \bar{\Psi}_{\tilde{q}}(\tilde{M}, \Gamma_{\text{int}}) \left[ \tilde{P}^{-1}(\tilde{q}^2) \left( \tilde{M}_{-\tilde{q}} + \bar{\Delta}_2 \frac{\delta \Gamma_{\text{int}}}{\delta \tilde{M}_{\tilde{q}}} \right) + \frac{\delta \Gamma_{\text{int}}}{\delta \tilde{M}_{\tilde{q}}} \right] \right. \\
& \left. - \frac{1}{1 + \bar{\Delta}_2 \Gamma_{\text{int}}^{(2)} [\tilde{q}; \tilde{M}]} \frac{\delta \bar{\Psi}_{\tilde{q}}(\tilde{M}, \Gamma_{\text{int}})}{\delta \tilde{M}_{\tilde{q}}} \right\} .
\end{aligned}$$

Thus (73) corresponds to:

$$\bar{\Psi}_{\tilde{q}}^{(P)}(\tilde{M}, \Gamma_{\text{int}}) = -\tilde{P}(\tilde{q}^2) \frac{\delta \Gamma_{\text{int}}}{\delta \tilde{M}_{-\tilde{q}}} .$$

One observes that the expression of the ERGE is complicated due to the explicit dependence on  $t$  carried by  $\bar{\Delta}_2$ . Usually one get rid of such an explicit dependence after completing the limit  $\Lambda_0 \rightarrow \infty$  at fixed  $\Lambda$ .

### 3.4.3 Limit of infinite overall cutoff

In the limit  $\Lambda_0 \rightarrow \infty$  the overall-cutoff function  $\Delta_0$  is commonly assumed to approach a regular form (according to the classical dimension (6) of  $\phi$ ):

$$\lim_{\Lambda_0 \rightarrow \infty} \tilde{\Delta}_0\left(\tilde{q}^2, \frac{\Lambda}{\Lambda_0}\right) = \frac{a}{(\tilde{q}^2)^{n_0}} ,$$

so that, keeping  $\Lambda$  and  $\mu$  fixed, we have:

$$\lim_{\Lambda_0 \rightarrow \infty} \bar{\Delta}_2(\tilde{q}^2, t) = \left[ Z_3\left(\frac{\mu}{\Lambda}\right) \right]^{-1} \frac{a}{(\tilde{q}^2)^{n_0}} - \tilde{P}(\tilde{q}^2) . \tag{75}$$



This result shows again a  $t$  dependence within  $\bar{\Delta}_2$  that illustrates perfectly our purposes in section 3.3.3 that  $\mu$  is an unacceptable lower limit to the running scale  $\Lambda$ . This is why we are reassured by the fact that the commonly authorized limit  $\Lambda_0 \rightarrow \infty$  induces in fact also the limit  $\mu \rightarrow 0$  (see section 3.3.2). Actually, the reasoning should be the reverse because to justify the recourse to an EMRO we should perform the limit  $\mu \rightarrow 0$  which, then, would induce the limit  $\Lambda_0 \rightarrow \infty$ . Indeed, the nature of the origin of  $\mu$  is different from that of  $\Lambda_0$ : the scale  $\mu$  appears for technical reasons and is linked to the particular way we have implemented RG-step 3 whereas  $\Lambda_0$  may have a physical origin that sometimes we would like to keep (not for field theoretical purposes however). It is interesting that the necessity of considering  $\mu \rightarrow 0$ , which seems to be inherent to the structural method, implies that  $\Lambda_0$  must go to infinity. But we are more accustomed to the process of sending  $\Lambda_0$  to infinity alone. So, to determine the value reached by  $Z_3\left(\frac{\mu}{\Lambda}\right)$  when  $\mu \rightarrow 0$ , let us recall the conditions under which we are commonly authorized to send  $\Lambda_0$  to infinity.

The main interest of the RG in field theory is that, on decreasing  $\Lambda$ , the RG-flow may approach eventual fixed points (or infra-red stable submanifolds) in the evolution-space of the actions. Such approaches in their final evolution are so slow and the RG-time value is so large that the initial  $\Lambda_0$  is very large compared to  $\Lambda$ . Moreover the precise form of  $\Delta_0$  has practically no effect anymore on the RG-flows in the vicinity of IR-stable manifolds, only the local topology of the RG trajectories in the space of actions is important. Then, provided that the RG-equation remains well defined, we may safely send  $\Lambda_0$  to infinity. Said in other words: only the close vicinity of IR-stable submanifolds are of interest in field theory so that one may get rid of both  $\mu$  and  $\Lambda_0$ . The knowledge of the resulting RG-flow equation is sufficient to determine the existence and the nature of the fixed points of interest for field theory. Consequently to determine the behavior of  $Z_3\left(\frac{\mu}{\Lambda}\right)$  in the limit  $\mu \rightarrow 0$  at fixed  $\Lambda$ , of interest for field theory, it is necessary to assume the vicinity of a fixed point.

According to (10, 9), near a fixed point and for  $n_0 = 0$ , we expect  $Z_3(\ell)$  to behave as:

$$Z_3(\ell) \sim \ell^{\eta^*-2},$$

so that we have:

$$Z_3(\ell) \xrightarrow{\ell \rightarrow 0} \infty,$$

provided that  $\eta^* < 2$ , which is the usual condition for a fixed point to be a “critical fixed point” [17].

Consequently, the limit  $\Lambda_0 \rightarrow \infty$  of (75) implies the condition:

$$\lim_{\Lambda_0 \rightarrow \infty} \bar{\Delta}_2(\tilde{q}^2, t) = \bar{\Delta}_2^\infty(\tilde{q}^2) = -\tilde{P}(\tilde{q}^2), \quad (76)$$

so that both (73) and (74) greatly simplify. However, after the limit of infinite cutoff  $\Lambda_0$  has been performed, it seems that the RG flow equation could display singularities due to the term  $1 - \tilde{P}(\tilde{q}^2) \Gamma_{\text{int}}^{(2)}[\tilde{M}]$  appearing in the denominator of a fraction. Taking benefit from the fact that the cutoff function is actually an integral part of the theory, this difficulty can be avoided if, as Ellwanger [3], we consider the RG flow equation for the *full* scale-dependent effective action  $\Gamma[\tilde{M}, t]$  prior to taking the limit (76).

### 3.5 Back to full actions

According to (39) in which the rescaling and the field-renormalization have been implemented (but not the limit  $\Lambda_0 \rightarrow \infty$  yet), the full effective action defines as:

$$\Gamma[\tilde{M}, t] = \frac{1}{2} \tilde{M} \cdot \bar{\Delta}_2^{-1} \cdot \tilde{M} + \Gamma_{\text{int}}[\tilde{M}, t] . \quad (77)$$

Then, we may consider safely the limit (76) within the RG-flow equation for  $\Gamma$  so that (74) and (73) leads respectively to (up to an additive constant):

$$\begin{aligned} \dot{\Gamma}[\tilde{M}, t] = & \int_q q^2 \frac{P'(q^2)}{\tilde{P}^2(\tilde{q}^2)} \left\{ \frac{1}{\Gamma^{(2)}[\tilde{q}]} + \tilde{M}_{\tilde{q}} \tilde{M}_{-\tilde{q}} \right\} \\ & + \mathcal{G}_{\text{dil}}(\Gamma, \tilde{M}, d_\phi^{(cw)}) + \varpi_0(t) \mathcal{O}_P^\infty(\Gamma, \tilde{M}) , \end{aligned} \quad (78)$$

$$\begin{aligned} \mathcal{O}_P^\infty(\Gamma, \tilde{M}) = & - \int_{\tilde{q}} \left\{ \tilde{P}^{-1}(\tilde{q}^2) \left[ \frac{1}{\Gamma^{(2)}[\tilde{q}; \tilde{M}]} + \tilde{M}_{\tilde{q}} \tilde{M}_{-\tilde{q}} \right] \right. \\ & \left. + \tilde{M}_{\tilde{q}} \frac{\delta \Gamma}{\delta \tilde{M}_{\tilde{q}}} + 1 \right\} . \end{aligned} \quad (79)$$

Because  $\tilde{P}(\tilde{q}^2)$  is assumed to decrease sufficiently rapidly toward zero when  $\tilde{q} \rightarrow \infty$ , a supplementary redefinition of the field  $\tilde{M}$  as (that, again, implies  $n_0 = 0$ ):

$$\tilde{M}_{\tilde{q}} = \tilde{P}(\tilde{q}^2) \tilde{\Phi}_{\tilde{q}} ,$$

eliminates possible bad behavior of the equation for large  $q$ , and we finally get:

$$\begin{aligned}
\dot{\Gamma} [\tilde{\Phi}, t] = & \int_q q^2 P' (q^2) \left( \frac{1}{\Gamma^{(2)} [\tilde{q}; \tilde{\Phi}]} + \tilde{\Phi}_{\tilde{q}} \tilde{\Phi}_{-\tilde{q}} \right) \\
& + \mathcal{G}_{\text{dil}} \left( \Gamma, \tilde{\Phi}, d_{\phi}^{(cw)} \right) + \varpi_0 (t) \bar{\mathcal{O}}_P^{\infty} \left( \Gamma, \tilde{\Phi} \right) \\
& + \int_q \left[ \mathbf{q} \cdot \frac{\partial \ln P (\tilde{q}^2)}{\partial \mathbf{q}} \right] \tilde{\Phi}_q \frac{\delta \Gamma}{\delta \tilde{\Phi}_q}, \tag{80}
\end{aligned}$$

$$\bar{\mathcal{O}}_P^{\infty} \left( \Gamma, \tilde{\Phi} \right) = - \int_{\tilde{q}} \left\{ \tilde{P} (\tilde{q}^2) \left[ \frac{1}{\Gamma^{(2)} [\tilde{q}; \tilde{\Phi}]} + \tilde{\Phi}_{\tilde{q}} \tilde{\Phi}_{-\tilde{q}} \right] + \tilde{\Phi}_{\tilde{q}} \frac{\delta \Gamma}{\delta \tilde{\Phi}_{\tilde{q}}} + 1 \right\}, \tag{81}$$

which, once  $\bar{\mathcal{O}}_P^{\infty}$  is split in two parts, also reads:

$$\begin{aligned}
\dot{\Gamma} [\tilde{\Phi}, t] = & \int_q \left[ -\varpi_0 (t) P (q^2) + q^2 P' (q^2) \right] \left\{ \frac{1}{\Gamma^{(2)} [\tilde{q}; \tilde{\Phi}]} + \tilde{\Phi}_{\tilde{q}} \tilde{\Phi}_{-\tilde{q}} \right\} \\
& + \mathcal{G}_{\text{dil}} \left( \Gamma, \tilde{M}, d_{\phi}^{(+)} \right) + \int_q \left[ \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} \ln |P| \right] \tilde{\Phi}_q \frac{\delta \Gamma}{\delta \tilde{\Phi}_q}. \tag{82}
\end{aligned}$$

Meanwhile the relation to  $S_{\text{int}} [\tilde{\phi}, t]$  has become:

$$S_{\text{int}} [\tilde{\phi}, t] = -\tilde{\phi} \cdot \tilde{\Phi} - \frac{1}{2} \tilde{\phi} \cdot P^{-1} \cdot \tilde{\phi} + \Gamma [\tilde{\Phi}, t], \tag{83}$$

$$P \cdot \tilde{\Phi} = -\tilde{\phi} + P \cdot \frac{\delta}{\delta \tilde{\phi}} S_{\text{int}} [\tilde{\phi}, t], \tag{84}$$

so that, due to the relation (76), the full action  $S [\tilde{\phi}, t]$  may now be reconstructed<sup>28</sup> to get the simple Legendre transformation (1, 2) stated in the introduction that with the properties:

$$\begin{aligned}
\dot{\Gamma} [\tilde{\Phi}, t] &= \dot{S} [\tilde{\phi}, t], \\
\mathcal{G}_{\text{dil}} \left( S, \tilde{\phi}, d_{\phi}^{(cw)} \right) &= \mathcal{G}_{\text{dil}} \left( \Gamma, \tilde{\Phi}, d_{\phi}^{(cw)} \right),
\end{aligned}$$

enables us to deduce (82) directly from the Wilsonian ERGE (54).

---

<sup>28</sup> Contrary to the common procedure where the infinite limit of the overall cutoff corresponded to (75) in which  $Z_3 \equiv 1$ .

## 4 Summary and final comments

After having presented a review of the various forms of ERGE currently encountered in the literature, with a particular attention given to the ways in which RG-step 3 (field-renormalization) has been realized, we have shown how an ERGE may be constructed in such a way that a simple Legendre transformation relates the RG-flow equation for the full scale dependent action  $S[\tilde{\phi}, t]$  to that of the full scale dependent effective action  $\Gamma[\tilde{\Phi}, t]$ . Though not new, the first important ingredient is the reference to Wilson's achievement of RG-step 1 via an "incomplete integration" in which no reference to any explicit UV-cutoff is required. Indeed, the number of degrees of freedom are reduced by integrating "more" the fields with large momenta than those with small momenta. Clearly, in this view there is no overall UV-scale  $\Lambda_0$  but only a pure momentum-scale of reference, at least in the RG-flow equation of the full action  $S$  (Wilsonian ERGE). Qualitatively, the situation may be compared to the perturbation field theory dimensionally regularized that avoids any consideration of explicit cutoff procedure but where a pure momentum scale automatically appears for dimensional reasons. Again not new, the second important ingredient is the systematic inclusion of the cutoff quadratic form into the full scale-dependent effective action  $\Gamma$ , that eliminates all the reference to any explicit cutoff procedure within the RG-flow equation. Then, provided the usual limit of infinite overall UV-cutoff  $\Lambda_0 \rightarrow \infty$  is undertaken, it only remains a formal running momentum scale (and eventual field-redefinitions to make the ERGEs well defined). Consequently the *full* actions no longer need to be affected by the introduction of arbitrary cutoff functions of any kind contrary to the common practice. It remains to link the two flow equations for the full actions, what was commonly thought to be impossible. To this regard, the most important (the third) ingredient is the global implementation of RG-step 3 in the ERGE, via the "operator" responsible for an infinitesimal change of normalization of the field by a pure constant (EMRO). Indeed with an EMRO, RG-step 3 may be seen as being obligatory "*equally implemented*" on all the fields  $\phi_q$  with  $0 \leq |q| < \infty$ . This leaves no room for any explicit cutoff. If, nevertheless, one insists to consider explicit cutoffs (IR and UV) and try to maintain the recourse to an EMRO, then RG-step 3 appears to be equally implemented only in the finite range  $\mu \leq |q| < \Lambda_0$  with  $\mu$  an effective overall IR cutoff linked to the usual overall UV-cutoff  $\Lambda_0$  so that  $\mu/\Lambda = \Lambda/\Lambda_0$  ( $\Lambda$  being the running momentum-scale). Then, the usual limit  $\Lambda_0 \rightarrow \infty$  of the common view is linked to the limit  $\mu \rightarrow 0$ . Using the common procedure that refers to the vicinity of a fixed point to justify the limit  $\Lambda_0 \rightarrow \infty$ , we show that, provided the anomalous dimension satisfies the condition  $\eta^* < 2$ , the limit  $\mu \rightarrow 0$  simplifies the usual relation between the UV and IR-cutoff functions so that a simple Legendre transformation links the two RG-flow equations.

The proof of existence of such a Legendre transformation has been given fol-

lowing the traditional way that first considers explicit cutoff functions so that the scale dependent actions  $S[\tilde{\phi}, t]$  and  $\Gamma[\tilde{\Phi}, t]$  have the usual definitions. In particular  $\Gamma[\tilde{\Phi}, t]$  may continue to be seen as interpolating between a “bare” action for small  $t$  and the “usual” effective action when  $t \rightarrow \infty$ . However, the main interest for field theory remains the vicinity of fixed points  $\Gamma^*[\tilde{\Phi}]$  which are a priori unknown prior to any explicit calculations.

We hope that the present work be of some help in the treatment of modern problems of field theory in which local symmetries (incompatible with an explicit cutoff procedure) are involved.

## A Field-renormalization and anomalous dimension

In [42], a confusion has been made between the field-renormalization function  $Z_3$  and its inverse. This is why we present again (see also [45]) the arguments that relate  $Z_3(\ell)$  to the anomalous dimension  $\eta^*$  –and by extension to  $\eta(t)$ – as indicated by (10, 9).

Provided one considers sufficiently large distances (small momenta), the original two-point correlation function:

$$G(|q_1|, S) \delta(q_1 + q_2) = \langle \phi_{q_1} \phi_{q_2} \rangle_S,$$

is preserved after renormalization. The decimation RG-step 1 does not modify the physics at large distances ( $q_1 \rightarrow 0$ ), then only the two remaining steps (rescaling and field-renormalization) can modify the property of the two point correlation function. Field-renormalization yields:

$$\langle \phi_{q_1} \phi_{q_2} \rangle_S = Z_3 \langle \phi_{q_1}^R \phi_{q_2}^R \rangle_{\bar{S}},$$

and the supplementary rescaling implies<sup>29</sup>:

$$\langle \tilde{\phi}_{\tilde{q}_1} \tilde{\phi}_{\tilde{q}_2} \rangle_S = Z_3 \ell^{2\bar{d}_\phi^{(c)}} \langle \tilde{\phi}_{\tilde{q}'_1}^R \tilde{\phi}_{\tilde{q}'_2}^R \rangle_{\bar{S}}, \quad (\text{A.1})$$

in which  $\bar{d}_\phi^{(c)}$  means  $d_\phi^{(c)} - d$  and  $d_\phi^{(c)}$  is the classical dimension of  $\phi(x)$  [defined by (6)].

(In the following of this appendix we assume that all quantities are dimensionless and we forget the tilde.)

---

<sup>29</sup> Roughly speaking we have  $|q| < \mu$  and after rescaling  $|q'| < \Lambda$  with  $\mu = \ell\Lambda$ , thus  $|q| = \ell|q'|$ , and by definition of  $\bar{d}_\phi^{(c)}$ ,  $\tilde{\phi}_{\ell\tilde{q}'_1} = \ell^{2\bar{d}_\phi^{(c)}} \tilde{\phi}_{\tilde{q}'_1}$ .

Expressed on  $G(q, S)$  –after having taken into account the dimension of the  $\delta$ -function– (A.1) becomes:

$$G(|q|, S) = Z_3 \ell^{2d_\phi^{(c)} - d} G(|q'|, S) . \quad (\text{A.2})$$

In general (A.2) is a complicated relation. But at a fixed point  $S^* = \bar{S}^*$  one expects a specific momentum dependence of the correlation function  $G^*(q) \equiv G(q, S^*)$ .

Indeed, knowing that  $|q| = \ell |q'|$  (see footnote 29), we get at a fixed point:

$$G^*(|q|) = Z_3 \ell^{2d_\phi^{(c)} - d} G^*\left(\frac{|q|}{\ell}\right) , \quad (\text{A.3})$$

which implies that  $G^*(|q|) \propto |q|^\alpha$ . This is precisely what is physically expected at a critical point with:

$$G(|q|) \underset{q \rightarrow 0}{\sim} G_0 |q|^{\eta^* - 2} , \quad (\text{A.4})$$

in which  $\eta^*$  is called the anomalous dimension of the field.

When reported in eq. (A.3) this gives:

$$Z_3 \ell^{2d_\phi^{(c)} - d} \ell^{2 - \eta^*} = 1 ,$$

and thus, taking into account (6):

$$Z_3 = \ell^{2(\eta^* - 1 + n_0)} , \quad (\text{A.5})$$

which is the fixed point expression of (10, 9) which, in turn, is nothing but an extension, away from any fixed point, of the power law (A.5).

## B Obtaining the expression of a simple EMRO

In this appendix all quantities are dimensionless.

With a view to use it in the structural method described in section 3.2, we aim at obtaining a simple expression of an EMRO associated to a given Wilsonian ERGE. To this end, we follow the procedure developped by O’Dwyer and Osborn in their appendix D of [15] with the difference that we deal with the full action  $S$  and a non-linear implementation of RG-step 3 instead of a “modified” Polchinski ERGE (for  $S_{\text{int}}$ ) and a RG-step 3 effected linearly (see section 2.3.3).

We start by considering a Wilsonian ERGE of the following general form:

$$\dot{S} = \int_q \left[ G(q^2) \left( \frac{\delta^2 S}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S}{\delta \phi_q} \frac{\delta S}{\delta \phi_{-q}} \right) + H(q^2) \phi_q \frac{\delta S}{\delta \phi_q} \right] + \mathcal{G}_{\text{dil}}(S, d_\phi), \quad (\text{B.1})$$

in which nothing is said on whether RG-step 3 has or has not been implemented. The question is to determine an EMRO  $\mathcal{O}_0(S^*, G, H, d_\phi)$  –i.e., a redundant operator with a zero eigenvalue– associated to that ERGE.

Let us consider the linearization of the ERGE (B.1) in the vicinity of a fixed point  $S^*$ , it leads to the eigenvalue equation:

$$\mathcal{D}\mathcal{U}^* = \lambda \mathcal{U}^*, \quad (\text{B.2})$$

with:

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4, \quad (\text{B.3})$$

$$\mathcal{D}_1 = \int_q \left[ (d - d_\phi) \phi_q + \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} \phi_q \right] \frac{\delta}{\delta \phi_q}, \quad (\text{B.4})$$

$$\mathcal{D}_2 = \int_q G(q^2) \frac{\delta^2}{\delta \phi_q \delta \phi_{-q}}, \quad (\text{B.5})$$

$$\mathcal{D}_3 = \int_q H(q^2) \phi_q \frac{\delta}{\delta \phi_q}, \quad (\text{B.6})$$

$$\mathcal{D}_4 = -2 \int_q G(q^2) \frac{\delta S^*}{\delta \phi_q} \frac{\delta}{\delta \phi_{-q}}. \quad (\text{B.7})$$

From the general form of a redundant operator:

$$\mathcal{O}_\Psi^* = \int_q \left[ \Psi_q \frac{\delta S^*}{\delta \phi_q} - \frac{\delta \Psi_q}{\delta \phi_q} \right], \quad (\text{B.8})$$

and the definitions (B.4–B.7), following similar calculations as those presented in appendix D of [15], one can show that:

$$\mathcal{D} \mathcal{O}_\Psi^* = \mathcal{O}_{\mathcal{D}_1 \Psi}^*, \quad (\text{B.9})$$

in which:

$$\mathcal{D}_1 \Psi_q = \mathcal{D} \Psi_q + \left[ (d_\phi - d - H(q^2)) - \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} \right] \Psi_q. \quad (\text{B.10})$$

As noted in [15] this result demonstrates that the operators  $\mathcal{O}_\Psi^*$  form a closed subspace under RG flow near a fixed point [9, 10].

In order to construct an EMRO, the idea of O'Dwyer and Osborn is to look for an operator  $\mathcal{O}_\Psi^*$  with:

$$\bar{\Psi}_q = Q(q^2) \mathcal{F}(q^2), \quad (\text{B.11})$$

in which  $\mathcal{F}(q^2)$  is assumed to have the property:

$$\mathcal{D} \mathcal{F}(q) = \left( \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} + \lambda_{\mathcal{F}} \right) \mathcal{F}(q). \quad (\text{B.12})$$

Consequently (B.10) gives:

$$\mathcal{D}_1 \bar{\Psi}_q = (d_\phi - d + \lambda_{\mathcal{F}}) \bar{\Psi}_q - \left( 2q^2 \frac{Q'}{Q} + H(q^2) \right) \bar{\Psi}_q,$$

so that choosing  $Q(x)$  as defined by:

$$2x \frac{Q'(x)}{Q(x)} = -H(x), \quad (\text{B.13})$$

implies that:

$$\mathcal{D}_1 \mathcal{O}_\Psi^* = (d_\phi - d + \lambda_{\mathcal{F}}) \mathcal{O}_\Psi^*,$$

and, owing to (B.9):

$$\mathcal{D} \mathcal{O}_\Psi^* = (d_\phi - d + \lambda_{\mathcal{F}}) \mathcal{O}_\Psi^*,$$

so that for:

$$\lambda_{\mathcal{F}} = d - d_\phi, \quad (\text{B.14})$$

$\mathcal{O}_\Psi^*$  is an EMRO.

To construct  $\mathcal{F}(q^2)$  with the required properties, O'Dwyer and Osborn propose the following form:

$$\mathcal{F}(q^2) = a(q^2) \phi_q + b(q^2) \frac{\delta S^*}{\delta \phi_{-q}}.$$

So that, using the properties –which follow from the definition of  $\mathcal{D}$  [eqs. (B.3, B.7)]:

$$\mathcal{D} \phi_q = \left( d - d_\phi + H(q^2) + \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} \right) \phi_q - 2G(q^2) \frac{\delta S^*}{\delta \phi_{-q}}, \quad (\text{B.15})$$

$$\mathcal{D} \frac{\delta S^*}{\delta \phi_q} = \left[ (d_\phi - H(q^2)) \frac{\delta}{\delta \phi_q} + \left( \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} \frac{\delta}{\delta \phi_q} \right) \right] S^*, \quad (\text{B.16})$$

it is not too complicated to show that the conditions (B.12–B.14) lead to the following coupled differential equations for  $a(x)$  and  $b(x)$  (in which the prime ' stands for  $\frac{d}{dx}$ ):



$$H(x) a(x) - 2xa'(x) = 0, \quad (\text{B.17})$$

$$2a(x) G(x) + b(x) (d - 2d_\phi) + H(x) b(x) + 2xb'(x) = 0. \quad (\text{B.18})$$

Actually, (B.17) is similar to (B.13) and we may thus set:

$$a(x) = \frac{A}{Q(x)}, \quad (\text{B.19})$$

in which  $A$  is some constant depending on initial conditions.

Then, if we introduce the function  $B(x)$ :

$$B(x) = -Q(x) b(x),$$

and choose  $A = 1$ , using (B.13) one may show that the differential equation (B.18) becomes [18]:

$$G(x) - \varpi_g B(x) - H(x) B(x) - xB'(x) = 0, \quad (\text{B.20})$$

$$\varpi_g = \frac{d}{2} - d_\phi. \quad (\text{B.21})$$

Assuming the initial condition:

$$B(0) = 1,$$

the solution of (B.20) then reads:

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ \left( \frac{\epsilon}{x} \right)^{\varpi_g} \frac{1}{C_\epsilon(x)} \left[ 1 + \epsilon^{-\varpi_g} \int_\epsilon^x u^{\varpi_g-1} G(u) C_\epsilon(u) du \right] \right\}, \quad (\text{B.22})$$

$$C_\epsilon(x) = e^{\int_\epsilon^x \frac{H(u)}{u} du}. \quad (\text{B.23})$$

Before discussing these expressions with explicit examples, let us come back to the differential equation (B.20).

Let us assume that we know a solution  $B_0(x)$  of (B.20) and that we use it to modify the ERGE (B.1) such that:

$$\begin{aligned} \dot{S} &\rightarrow \dot{S} + \alpha \mathcal{O}_{\bar{\Psi}_0}, \\ \bar{\Psi}_{0,q} &= \phi_q - B_0(q^2) \frac{\delta S^*}{\delta \phi_{-q}}, \end{aligned}$$

then (B.20) is changed into a differential equation for  $B_\alpha(x)$  which reads:

$$0 = G(x) + \alpha [B_0(x) - B_\alpha(x)] - \varpi_g B_\alpha(x) - H(x) B_\alpha(x) - x B'_\alpha(x) ,$$

so that  $B_\alpha(x) = B_0(x)$  is still a solution. Consequently, an EMRO like  $\mathcal{O}_{\bar{\Psi}_0}$  (corresponding to  $\varpi_g = 0$ ) may be used to implement RG-step 3 without altering its quality of EMRO.

Let us illustrate this on two examples on which we shall show that the initial condition  $B_\alpha(0) = \text{const.}$ , imposed by the requirement of quasi-locality for  $S$ , induces the condition  $\eta^* < 2$ .

The two examples correspond to a choice of cutoff function (19) with, respectively,  $n_0 = 0$  (Wilson's choice) and  $n_0 = 1$  (Polchinski's choice).

### B.1 Wilson's choice

The Wilson ERGE extended to an arbitrary cutoff function  $P(q^2)$  as given by (19) with  $n_0 = 0$  and prior to realization of RG-step 3, corresponds to (B.1) with the following choices [see eqs. (54, 7) with  $\varpi_0 = 0$ ]:

$$\begin{aligned} G(x) &= -x P'(x) , \\ H(x) &= -2x \frac{P'(x)}{P(x)} , \\ d_\phi &= d_\phi^{(cw)} = \frac{d}{2} , \\ \varpi_g &= 0 , \end{aligned}$$

and the solution (B.22, B.23) reads:

$$C_\epsilon(x) = \left( \frac{P(\epsilon)}{P(x)} \right)^2 , \tag{B.24}$$

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ P^2(x) \left[ \frac{1}{P^2(\epsilon)} - \int_\epsilon^x \frac{P'(u)}{P^2(u)} du \right] \right\} , \tag{B.25}$$

which, after integration and provided that  $P(0) = 1$ , give:

$$B(x) = P(x) .$$

The second step, consists in considering the case  $\varpi = \varpi_0 = 1 - \eta^*/2$  in (B.1). That amounts to finding the solution of (B.20) in which the following changes are effected:

$$G(x) \rightarrow G(x) + \varpi_0 B(x) ,$$

$$\varpi_g \rightarrow \varpi_g + \varpi_0 ,$$

in that case (B.24) is unchanged and (B.25) is replaced by:

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ \left( \frac{\epsilon}{x} \right)^{\varpi_0} P^2(x) \left[ \frac{1}{P^2(\epsilon)} + \varpi_0 \epsilon^{-\varpi_0} \int_{\epsilon}^x u^{\varpi_0-1} \frac{B(u)}{P^2(u)} du \right. \right. \\ \left. \left. - \epsilon^{-\varpi_0} \int_{\epsilon}^x u^{\varpi_0} \frac{P'(u)}{P^2(u)} du \right] \right\} .$$

By integrating by parts the last term, it comes, after some rearrangement:

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ P(x) + \left( \frac{\epsilon}{x} \right)^{\varpi_0} P^2(x) \left( \frac{1}{P^2(\epsilon)} - \frac{1}{P(\epsilon)} \right) \right. \\ \left. + \left( \frac{1}{x} \right)^{\varpi_0} P^2(x) \varpi_0 \int_{\epsilon}^x u^{\varpi_0-1} \left( \frac{B(u)}{P^2(u)} - \frac{1}{P(u)} \right) du \right\} ,$$

in which the second term gives 0 in the limit  $\epsilon \rightarrow 0$  provided that  $\varpi_0 > 0$  (and  $P(0) = \text{const}$ ) and the second term vanishes if  $B(x) = P(x)$  which is actually the case, what confirms (55).

We note that the condition  $\varpi_0 > 0$  corresponds to the usual condition  $\eta^* < 2$  to have a “critical fixed point” (see [17]).

## B.2 Polchinski's choice

The Polchinski choice for the arbitrary cutoff function  $P(q^2)$  is (19) with  $n_0 = 1$ . This choice, which is perfectly acceptable, corresponds to (B.1) with:

$$G(x) = -K'(x) ,$$

$$H(x) = -2x \frac{K'(x)}{K(x)} ,$$

$$d_\phi = d_\phi^{(c)} = \frac{d}{2} - 1 ,$$

$$\varpi_g = 1 ,$$

and the solution (B.22, B.23) reads:

$$C_\epsilon(x) = \left( \frac{K(\epsilon)}{K(x)} \right)^2, \quad (\text{B.26})$$

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ \frac{K^2(x)}{x} \left[ \frac{\epsilon}{K^2(\epsilon)} - \int_\epsilon^x \frac{K'(u)}{K^2(u)} du \right] \right\}, \quad (\text{B.27})$$

then, after integration, we get in the limit  $\epsilon \rightarrow 0$  (and provided that  $K(0) = 1$ )

$$B(x) = \frac{K(x)}{x} [1 - K(x)]. \quad (\text{B.28})$$

The second step, consists in considering the case  $\varpi = \varpi_1 = -\eta^*/2$  in (B.1) that corresponds to setting  $\varpi_g = 1 - \eta^*/2 = \varpi_0$  and  $G(x) \rightarrow G(x) + \varpi_1 B(x)$ . As previously, (B.26) is unchanged and (B.27) is replaced by:

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ \left( \frac{\epsilon}{x} \right)^{\varpi_0} K^2(x) \left[ \frac{1}{K^2(\epsilon)} + \varpi_1 \epsilon^{-\varpi_0} \int_\epsilon^x u^{\varpi_0-1} \frac{B(u)}{K^2(u)} du - \epsilon^{-\varpi_0} \int_\epsilon^x u^{\varpi_0-1} \frac{K'(u)}{K^2(u)} du \right] \right\},$$

By integrating by parts the last term, it comes, after some rearrangement (using the relation  $\varpi_1 = \varpi_0 - 1$ ):

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ \frac{K(x)}{x} + \left( \frac{\epsilon}{x} \right)^{\varpi_0} K^2(x) \left( \frac{1}{K^2(\epsilon)} - \frac{1}{\epsilon K(\epsilon)} \right) + \varpi_1 \frac{K^2(x)}{x^{\varpi_0}} \int_\epsilon^x u^{\varpi_0-1} \left( \frac{B(u)}{K^2(u)} - \frac{1}{u K(u)} \right) du \right\},$$

if one uses (B.28) in the r.h.s. then the last term simplifies and we get:

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ \frac{K(x)}{x} + \left( \frac{\epsilon}{x} \right)^{\varpi_0} K^2(x) \left( \frac{1}{K^2(\epsilon)} - \frac{1}{\epsilon K(\epsilon)} \right) - \varpi_1 \frac{K^2(x)}{x^{\varpi_0}} \int_\epsilon^x u^{\varpi_0-2} du \right\},$$

which, after integration gives:

$$B(x) = \lim_{\epsilon \rightarrow 0} \left\{ \left[ \frac{K(x)}{x} (1 - K(x)) + \left( \frac{\epsilon}{x} \right)^{\varpi_0} K^2(x) \left( \frac{1}{K^2(\epsilon)} - \frac{1}{\epsilon K(\epsilon)} + \frac{1}{\epsilon} \right) \right] \right\},$$

and the last term vanishes in the limit  $\epsilon \rightarrow 0$  provided that  $K(0) = 1$  and  $\varpi_0 > 0$  (i.e.  $\eta^* > 2$ ). We thus find (B.28).

We note that the EMRO is not as simple as in the preceding case but, once extended out of the fixed point, it may well be used to implement RG-step 3 within an ERGE to give, instead of (28) for  $n_0 = 1$ :

$$\begin{aligned} \dot{S}[\phi, t] = & - \int_q K'(q^2) \left[ \frac{\delta^2 S}{\delta\phi_q \delta\phi_{-q}} - \frac{\delta S}{\delta\phi_q} \frac{\delta S}{\delta\phi_{-q}} + 2q^2 K^{-1} \phi_q \frac{\delta S}{\delta\phi_q} \right] \\ & + \mathcal{G}_{\text{dil}}(S, \phi, d_\phi^{(c)}) + \varpi_1(t) \mathcal{O}_K, \end{aligned} \quad (\text{B.29})$$

in which:

$$\begin{aligned} d_\phi^{(c)} &= \frac{d-2}{2}, \quad \varpi_1(t) = -\frac{\eta(t)}{2}, \quad K(0) = 1, \\ \mathcal{O}_K &= \int_q \left[ \frac{K(q^2)}{q^2} [1 - K(q^2)] \left( \frac{\delta^2 S}{\delta\phi_q \delta\phi_{-q}} - \frac{\delta S}{\delta\phi_q} \frac{\delta S}{\delta\phi_{-q}} \right) + \phi_q \frac{\delta S}{\delta\phi_q} \right]. \end{aligned}$$

$\mathcal{O}_K^*$  being an EMRO for (B.29).

## References

- [1] K.G. Wilson, J. Kogut, Phys. Rep. 12 (1974) 75.
- [2] C. Wetterich, Phys. Lett. B 301 (1993) 90 .
- [3] U. Ellwanger, Z. Phys. C 62 (1994) 503. [arXiv:hep-ph/9308260]
- [4] T. R. Morris, Int. J. Mod. Phys. A 09 (1994) 2411. [arXiv:hep-ph/9308265]
- [5] M. D’Attanasio and T. R. Morris, Phys. Lett. B 409 (1997) 363. [arXiv:hep-th/9704094]
- [6] H. Osborn and D. E. Twigg, Annals Phys. 327 (2012) 29. [arXiv:1108.5340]
- [7] O. J. Rosten, “*Relationships Between Exact RGs and some Comments on Asymptotic Safety*”, unpublished (2011). [arXiv:1106.2544]
- [8] O. J. Rosten, J. Phys. A: Math. Theor. 44 (2011) 195401. [arXiv:1010.1530]
- [9] F. J. Wegner, J. Phys. C: Solid State Phys. 7 (1974) 2098.
- [10] F. J. Wegner, “*The critical state, General aspects*”, in Phase Transitions and Critical Phenomena Vol. VI, p. 7, *Ed. by C. Domb and M.S. Green* (Acad. Press, N.-Y., 1976).
- [11] J. I. Latorre and T. R. Morris, J. High Energy Phys. 11 (2000) 004. [arXiv:hep-th/0008123]

- [12] C. Bagnuls and C. Bervillier, Phys. Rep. 348 (2001) 91. [arXiv:hep-th/0002034]
- [13] J. Polchinski, Nucl. Phys. B 231 (1984) 269.
- [14] R. D. Ball, P. E. Haagensen, J. I. Latorre and E. Moreno, Phys. Lett. B 347 (1995) 80. [arXiv:hep-th/9411122]
- [15] J. P. O’Dwyer and H. Osborn, Annals Phys. 323 (2008) 1859. [arXiv:0708.2697v2]
- [16] J. Berges, N. Tetradis and C. Wetterich, Phys. Rep. 363 (2002) 223. [arXiv:hep-ph/0005122]
- [17] O. J. Rosten, Phys. Rep. 511 (2012) 177. [arXiv:1003.1366]
- [18] C. Bervillier, Cond. Matt. Phys. 16 (2013) 23003. [arXiv:1304.4131]
- [19] T. R. Morris, Prog. Theor. Phys. Suppl. 131 (1998) 395. [arXiv:hep-th/9802039]
- [20] K. I. Aoki, Int. J. Mod. Phys. B 14 (2000) 1249.
- [21] M. Salmhofer and C. Honerkamp, Prog. Theor. Phys. 105 (2001) 1.
- [22] J. Polonyi, Cent. Eur. J. Phys. 1 (2003) 1. [arXiv:hep-th/0110026 ]
- [23] B. Delamotte, D. Mouhanna and M. Tissier, Phys. Rev. B 69 (2004) 134413. [arXiv:cond-mat/0309101]
- [24] J. M. Pawłowski, Ann. Phys. (N.Y.) 322 (2007) 2831. [arXiv:hep-th/0512261]
- [25] B. Delamotte, in “*Order, Disorder and Criticality. Advanced Problems of Phase Transition Theory, Vol 2*”, p. 1, Ed. by Yu. Holovatch (World Scientific, Publ. Co., Singapore, 2007); also in Lect. Notes Phys. 852 (2012) 49. [arXiv:cond-mat/0702365]
- [26] B.-J. Schaefer and J. Wambach, Phys. Part. Nucl. 39 (2008) 1025. [arXiv:hep-ph/0611191]
- [27] Y. Igarashi, K. Itoh and H. Sonoda, Prog. Theor. Phys. Suppl. 181 (2009) 1. [arXiv:0909.0327]
- [28] P. Kopietz, L. Bartosch and F. Schütz, Lect. Notes Phys. 798 (2010).
- [29] J.-P. Blaizot, Lect. Notes Phys. 852 (2012) 1.
- [30] H. Gies, Lect. Notes Phys. 852 (2012) 287. [arXiv:hep-ph/0611146]
- [31] S. Nagy, “*Lectures on renormalization and asymptotic safety*”, unpublished (2012). [arXiv:1211.4151]
- [32] W. Metzner, M. Salmhofer, C. Honerkamp, V. Meden and K. Schoenhammer, Rev. Mod. Phys. 84 (2012) 299. [arXiv:1105.5289]
- [33] J. Braun, J. Phys. G: Nucl. Part. Phys. 39 (2012) 033001. [arXiv:1108.4449]
- [34] A. Wipf, Lect. Notes Phys. 864 (2013) 257.

- [35] C. Platt, W. Hanke and R. Thomale, Adv. in Phys. 62 (2013) 453.  
[arXiv:1310.6191]
- [36] T. L. Bell and K. G. Wilson, Phys. Rev. B 11 (1975) 3431.
- [37] E. K. Riedel, G. R. Golner and K. E. Newman, Annals Phys. 161 (1985) 178.
- [38] G. R. Golner, Phys. Rev. B 33 (1986) 7863.
- [39] T. R. Morris, Phys. Lett. B 329 (1994) 241. [arXiv:hep-ph/9403340]
- [40] M. Bonini, M. D’Attanasio and G. Marchesini, Nucl. Phys. B 409 (1993) 441.  
[arXiv:hep-th/9301114]
- [41] J. Zinn-Justin, in “*Quantum Field Theory and Critical Phenomena*”, “*First edition*” (Clarendon Press, Oxford, 1989).
- [42] C. Bervillier, Phys. Lett. A 332 (2004) 93. [arXiv:hep-th/0405025]
- [43] B. J. Warr, Annals Phys. 183 (1988) 1.
- [44] R. D. Ball and R. S. Thorne, Annals Phys. 236 (1994) 117. [arXiv:hep-th/9310042]
- [45] M. E. Fisher, Lect. Notes Phys. 186 (1983) 1.